Bowen's equation & multifractal analysis

§1
Linear Cantor sets & Moran's equation

§2 Non-linear Cantor sets & Bowen's equation

Thermodynamic formalism & classical results

Dimension theory & new results

Multifractal analysis

Birkhoff & Lyapunov spectra - define

Legendre transforms

Other spectra & general picture

§0
Hausdorff dim: \( Z \subset \mathbb{R}^d \), \( \varepsilon > 0 \)

\[ D(Z, \varepsilon) = \left\{ \{x_i, r_i; i \in I | x_i \in Z, 0 < r_i < \varepsilon, U_i \cap B(x_i, r_i) \supset Z \} \right\} \]

(set of covers)

\[ m_h(Z, \alpha, \varepsilon) = \inf \sum_i r_i^\alpha \] (outer measure)

\[ \forall \alpha > 0 \]

\[ m_f(Z, \alpha) = \lim_{\varepsilon \to 0} m_h(Z, \alpha, \varepsilon) \]

\[ \dim_h Z = \inf \alpha \mid m_f(Z, \alpha) = 0 \]

\[ \dim_f Z = \sup \alpha \mid m_f(Z, \alpha) = \infty \]

Box dim: \( S(Z, \varepsilon) \) a minimal \( \varepsilon \)-spanning set,

\[ \dim_B Z = \lim_{\varepsilon \to 0} \frac{\log \# S(Z, \varepsilon)}{\log (1/\varepsilon)} \]

1 Canonical example: middle-third Cantor set

This is Hausdorff dimension as well:

\[ \sigma_1, \sigma_2 \text{ contractions by } \frac{1}{3}, \quad J = \sigma, J U \sigma_2 J \]

\[ m_f(J, \alpha) = m_f(\sigma, J, \alpha) + m_f(\sigma_2 J, \alpha) \]

\[ = (\frac{1}{3})^\alpha m_f(J, \alpha) + (\frac{1}{3})^\alpha m_f(J, \alpha) = 2 (\frac{1}{3})^\alpha m_f(J, \alpha) \]

2 Tweak a bit: \( \sigma_1, \sigma_2 \) contractions by \( \rho_1, \rho_2 \in (0, 1) \)

\[ m_f(J, \alpha) = \rho_1^\alpha m_f(J, \alpha) + \rho_2^\alpha m_f(J, \alpha) \]

\[ \Rightarrow \rho_1^\alpha + \rho_2^\alpha = 1 \] (Moran's equation)

in general, \( \rho_1^t + \ldots + \rho_k^t = 1 \)

Box dim not so obvious

\( \Rightarrow t = \dim_h J \)
§1

Write Morse's equation in a funny way:

Pick $x_j \in I_j$, then $p_j = \frac{1}{f''(x_j)} = e^{-\log f''(x_j)}$

$\Rightarrow (\ast)$ becomes $\sum_j e^{-t \log f''(x_j)} = 1$

* Non-linear $\Rightarrow f'(x_j)$ no longer gives $p_j$, but, asymptotic behaviours are related.

$I_j^n(x) = \{ y \mid f^k(x), f^k(y) \text{ in same } I_j \forall 0 \leq k \leq n \}$

$|I_j^n(x)| \approx \left| \frac{f''(x_j)}{f''(x)} \right|^{-1} = \left| f'(x) f'(f(x)) \ldots f'(f^{n-1}(x)) \right|^{-1}$

$= e^{-\sum_n (\log f'(x))}$

where $\sum_n \varphi(x) = \varphi(x) + \varphi(f(x)) + \ldots + \varphi(f^{n-1}(x))$

* What did $\sum_j p_j = 1$ give? Allowed us to say that

$\sum_{x \in E_n} |I_j^n(x)|^t = 1$, where $E_n$ includes 1 pt from each $I_j^n$.

The sum is $ \approx m_h(\mathcal{J}, t, \varepsilon_n)$, and corresponds to

$\sum_{x \in E_n} e^{S_n \varphi(x)}$, where $\varphi = -t \log |f'|$

* $E(n, \varepsilon)$ a minimal $(n, \varepsilon)$-spanning set:

$\cup_{x \in E} B(x, n, \varepsilon) \supset \mathcal{J}$, $B(x, n, \varepsilon)$ = Bowen ball

$Z(n, \varepsilon) = \Sigma E e^{S_n \varphi(x)}$ = partition

topological pressure = $p(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log Z(n, \varepsilon)$

* $\Sigma_{x \in E_n} e^{S_n (-t \log |f'|)(x)}$ remains odd for $t = \dim_h \mathcal{J}$

$\Rightarrow p(-t \log |f'|) = 0 \quad (\ast \ast)$ (Bowen's equation)
**Thm** (Ruelle, 1982)

\[ M, \text{ a Riem. mfd, } V \subset M \text{ open, } f : V \to M, C^{1+\varepsilon} \]. Suppose

1. \( f \) conformal: \( Df(x) = a(x) I \) for all \( x \in V \)
2. \( J \) compact & invariant
3. \( J \) maximal: \( J = \{ x \in V \mid f^n(x) \in V \text{ for some } n \geq 0 \} \)
4. \( f \) top. mix on \( J \)
5. \( f \) unif. expanding on \( J \): \( \exists r > 1 \) s.t. \( a(x) \geq r \) for all \( x \).

Then \( \exists t \in \mathbb{R} \) s.t. \( P_f (-t \log a) = 0 \), and for this \( t \),
we have \( \dim_J f^t = t \). (1997 \( \rightarrow \) C by Iturriaga, Pesin)

**Q:** What if we are interested in non-compact subsets of \( J \)?

**Eq**

\[
I_n = \mathbb{Z}, \quad a_n(x) = \# \{ k \in \mathbb{N} \mid f^k(x) \in I_n \}
\]

\[ Z = \{ x \mid \frac{a_n(x)}{n} \to \frac{1}{2} \} \]

\((\mu(Z) = 1, \mu = m e)

**What is \( \dim_H Z \)?**

Want to compute \( \dim_H Z \) but need proper defn of pressure.

**Usual defn of pressure corresponds to box dim -- replace \( B(x, \varepsilon) \) with \( B(x, n, \delta) \).** assign weights by \( q \).

**Pesin \& Pollicott gave defn as dimension -- ref to B pt:**

\[
P(Z, N, \delta) = \sum \{ (x, n, \delta) \in I \mid x \in Z, n \geq N, U \subset B(x, n, \delta) \}
\]

\[ m_p(Z, q, t, N, \delta) = \inf \sum_{x \in I} e^{-t \cdot q(x)} \]

\[ m_p(Z, q, \tau, \delta) = \lim_{N \to \infty} m_p(Z, q, t, N, \delta) \]

\[ P_Z(q, \delta) = \lim_{\delta \to 0} P_{Z}(q, \delta) \]

**Barreira \& Schmeling (2000) used this to remove requirement 2 above.**

**What if uniform expansion fails?**

\[ Z \text{ as before, still have } \lambda(x) > 0 \text{ on } Z \ldots \]
Thm (C., 2009)

Let $X$ be a cusp metric space. A conformal map $f: X \to Y$ has no critical points if

$$a(x) = \lim_{y \to x} \frac{d(f(x), f(y))}{d(x, y)}$$

exists and is finite, non-zero.

Define

$$\lambda(x) = \lim_{n \to \infty} \frac{1}{n} S_n(\log \alpha)(x), \quad \overline{\lambda}(x)$$

for $Z \subset X$ be such that for every $x \in Z$,

1. $0 < \lambda(x) \leq \overline{\lambda}(x) < \infty$
2. Either $\lambda(x) = \overline{\lambda}(x)$ or $\inf \left\{ S_n(\log \alpha)(f^k(x)) \mid n, k \geq 1 \right\} = -\infty$.

Then

$$\dim_H Z = \inf \{ t \mid P_z(-t \log \alpha) \leq 0 \}$$

is such that

$$\sup \{ t \mid P_z(-t \log \alpha) > 0 \}$$

Remark. Suppose $\lambda(x) = \lim_{n \to \infty} \frac{1}{n} S_n(\log \alpha) = \lambda \in \mathbb{R}^+$ and $x \in Z$.

Then one may show that $P_z(-t \log \alpha) = P_z(0) - t \lambda$.

and since $P_z(0) = h_{top} Z$, we have

$$\dim_H Z = \frac{h_{top} Z}{\lambda}$$

* For an application of this, we turn to multifractal analysis.

§2

Birkhoff Ergodic Thm

If $\mu$ is ergodic, $q \in L'_{\mu}$, then $\frac{1}{n} S_n q(x) \to \int q \, d\mu \mu$-a.e.

But, of course, the limit may take many different values, and different measures see different limits.

Ex: $q = \chi_I$, in previous example, $S_n q(x) = a_n(x)$

Easy to construct pts with any limit in $(0, 1]$.

Define

$$K_\alpha^q = \{ x \mid \frac{1}{n} S_n q(x) \to \alpha \}$$

$$X = \bigcup_{\alpha \in R} K_\alpha^q \cup X'$$

multifractal decomposition

Q: How big are the various level sets?
* How to quantify $K^q_α$? Measures useless. So use a dimensional quantity - but they are clumps, so can't use box dim. Thus, use $dim_H$, or $h_{top}$.

**Def.** The Birkhoff spectrum of $q$ is

$$B(α) = h_{top} K^q_α$$

**Multifractal miracle** $B(α)$ is often smooth & concave!

Why? Legendre transform:

1. $$B(α) = \inf_{t \in \mathbb{R}} (P_x(tα) - tα)$$
2. $$P_x(tα) = \sup_{α \in \mathbb{R}} (B(α) + tα)$$

* If (1) & $t \cdot P_x(tα)$ smooth & strictly convex, then $B(α)$ smooth & strictly convex.
* To get (1), first get (2) !

2 incorrect arguments

1. $$P_x(tα) = \sup_{α \in \mathbb{R}} P_{K^q_α}(tα) = \sup_{α \in \mathbb{R}} (P_{K^q_α}(0) + tα)$$
   $$= \sup_{α \in \mathbb{R}} (B(α) + tα)$$

2. $$P_x(tα) = \sup_{μ \in \mathcal{M}} (h_{μ}(tφ) + tα)$$
   $$= \sup_{α \in \mathbb{R}} \sup_{μ \in \mathcal{M}} (h_{μ}(tφ) + tα)$$
   $$= \sup_{α \in \mathbb{R}} [\left( \sup_{μ \in \mathcal{M}} h_{μ}(tφ) \right) + tα]$$
   $$= \sup_{α \in \mathbb{R}} [h_{top} K^q_α + tα] = \sup_{α \in \mathbb{R}} (B(α) + tα)$$

* Fortunately, 2 wrongs make a right - each of these gives one inequality.
Thm (C., 2009)

\[ X \ni \phi, \ f: X \ni \to, \ \varphi: X \to \mathbb{R} \to. \]

Suppose that for \( \varphi \in (\varphi_1, \varphi_2) \) we have

I. \( \exists \) an equilibrium state \( \mu_{\varphi} \) for \( \varphi \)

II. \( \varphi \to p(\varphi) \) differentiable at \( \varphi \)

(or \( \mu_{\varphi} \) is unique)

Then \( \varphi \to p(\varphi) \) is log. transform of \( B(\alpha) \)

B (if \( (\ast) \), then let \( x_i = \frac{d}{d\varphi} p(\varphi) \mid_{\varphi = \varphi_i} \), and

\[ B(\alpha) = \inf_{\alpha \in \mathbb{R}} \left( p(\alpha) - q_\alpha \right) \]

\[ \forall \alpha \in (\alpha_1, \alpha_2) \]

Thus \( B \) strictly concave & \( C^1 \) except at \( \alpha \)

so \( p(\varphi) \) affine.

Special case - \( \varphi = \log|Df| \) is geometric potential.

* Get Lyapunov spectrum \( \text{htop} K_{\alpha} \)

* \( \lambda(x) \) constant on \( K_{\alpha} \log|Df| \) \( \to \) gen. Bowen's ap. applies

if \( f \) conformal w/o critical pts:

\[ \dim_K \log|Df| = \frac{1}{\alpha} \text{htop} K_{\alpha} \log|Df| \]

\[ = \frac{1}{\alpha} \inf_{\alpha \in \mathbb{R}} \left[ p(\alpha \log|Df|) - q_\alpha \right] \]

Other multifractal spectra:

* 2 ingredients:
  1. local quantity - Birkhoff, Lyapunov, entropy, dimension
  2. dimensional quantifier - entropy or H. dim.

* map pressure \( \to \) Birkhoff \( \to \) Gibbs \( \to \) entropy

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