Thermodynamics with small obstructions

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November 15, 2011

Joint work with Daniel J. Thompson (Penn State)
specification $\Rightarrow$ intrinsic ergodicity (unique MME)

Theorem (C.–Thompson)

If all obstructions to specification have small entropy, then the MME is unique.
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Theorem (C.–Thompson)

If all obstructions to specification have small entropy, then the MME is unique.

specification + Bowen property $\Rightarrow$ unique equilibrium state

Theorem (C.–Thompson)

If all obstructions to specification and the Bowen property have small pressure, then the equilibrium state is unique.
Topological pressure

Topological dynamical system:
- $X$ a compact metric space, $f : X \to X$ continuous
- $\mathcal{M} = \{\text{Borel } f\text{-invariant probability measures on } X\}$

Variational principle 1: $h_{\text{top}}(X, f) = \sup_{\mu \in \mathcal{M}} h_\mu(f)$
Variational principle 2: $P(\varphi) = \sup_{\mu \in \mathcal{M}} \left( h_\mu(f) + \int \varphi \, d\mu \right)$
Maximum achieved by MME/equilibrium state
Topological pressure

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Maximum achieved by MME/equilibrium state

Example

$X = \Sigma_2^+ = \{0, 1\}^\mathbb{N}$, $p, q \in \mathbb{R}$, $\varphi(x) = p1_{[0]} + q1_{[1]}$

Then $P(\varphi) = \log(e^p + e^q)$ and the unique equilibrium state is $(\alpha, 1 - \alpha)$-Bernoulli, where $\alpha = \frac{e^p}{e^p + e^q}$.

When is there a unique equilibrium state?
Thermodynamic formalism

Topological pressure $P(\varphi) = \sup_{\mu}(h_\mu(f) + \int \varphi \, d\mu)$:

- supremum of affine functions $\Rightarrow$ convex function $C(X) \to \mathbb{R}$;
- equilibrium states are tangent functionals in $C(X)^*$.

If entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous, then:

- so is $\mu \mapsto (h_\mu(f) + \int \varphi \, d\mu) \Rightarrow$ existence;
- unique $\iff$ $P$ (Gâteaux) differentiable;
- convex $\Rightarrow$ diff. on residual set $\Rightarrow$ unique.
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This says nothing about which potentials admit unique equilibrium states, or what the ergodic properties of those states are.

**General principle:** Uniform mixing of $X$ and regularity of $\varphi$ should imply uniqueness and strong ergodic properties.
Focus on shift spaces (subshifts):
- \( X \subset \Sigma_\sigma^+ \) closed and \( \sigma \)-invariant, NOT NECESSARILY AN SFT
- language of \( X \): \( \mathcal{L} = \mathcal{L}(X) = \{x_1 \cdots x_n \mid x \in X, n \geq 1\} \)
- words of length \( n \): \( \mathcal{L}_n = \{w \in \mathcal{L} \mid |w| = n\} \)
- entropy: \( h_{\text{top}}(X, \sigma) = \lim_{n \to \infty} \frac{1}{n} \log \# \mathcal{L}_n \)
- pressure: \( P(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{w \in \mathcal{L}_n} e^{\sup_{x \in [w]} S_n \varphi(x)} \)
- \( S_n \varphi(x) = \varphi(x) + \varphi(\sigma x) + \cdots + \varphi(\sigma^{n-1} x) \)

expansive \( \Rightarrow \) entropy map \( \mathcal{M} \to \mathbb{R} \) upper semi-continuous

Existence guaranteed: the real question is uniqueness.

Beyond transitivity, what do we need?
Uniqueness may fail

Example

Let $X \subset \Sigma_5^+ = \{0, 1, 2, 1, 2\}^\mathbb{N}$ be the shift whose language $\mathcal{L}$ contains $v0^nw$ and $w0^nv$ if and only if $n \geq 2 \max(|v|, |w|)$.

- $(X, \sigma)$ is topologically transitive (indeed, mixing)
- $h_{top}(X, \sigma) = \log 2$
- 2 measures of maximal entropy:
  \[ \nu = (\frac{1}{2}, \frac{1}{2})\text{-Bernoulli on } \{1, 2\}^\mathbb{N}, \]
  \[ \mu = (\frac{1}{2}, \frac{1}{2})\text{-Bernoulli on } \{1, 2\}^\mathbb{N}. \]

Uniqueness of an MME can fail for transitive shifts.
Classes of intrinsically ergodic shifts

The following are intrinsically ergodic (unique MME):
- Irreducible subshifts of finite type (Parry 1964)
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- Irreducible subshifts of finite type (Parry 1964)
- Shifts with specification (Bowen 1974)
- $\beta$-shifts (Walters 1978, Hofbauer 1979)
The motivating question

Intrinsic ergodicity is not necessarily preserved by factors.

- $X \subset \{0, 1, 2, 1, 2\}^\mathbb{N}$ as before: two MMEs
- $Y \subset \Sigma_6^+ = \{0, 1, 2, 1, 2, 3\}^\mathbb{N}$: one MME
- $\pi: Y \to X$ by $\pi(3) = 2$, others unchanged

Specification is preserved by factors, so intrinsic ergodicity survives.

What about $\beta$-shifts? (Klaus Thomsen, Mike Boyle)
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What about \( \beta \)-shifts? (Klaus Thomsen, Mike Boyle)

**Theorem (C.–Thompson 2010)**

\((X, \sigma)\) a subshift factor of a \( \beta \)-shift

\[ \Rightarrow \text{all obstructions to specification have zero entropy} \]

\[ \Rightarrow (X, \sigma) \text{ is intrinsically ergodic} \]
For $\beta > 1$, $\Sigma_\beta$ is the coding space for the map

$$f_\beta : [0, 1] \to [0, 1], \quad x \mapsto \beta x \pmod{1}$$

$$1_\beta = a_1 a_2 \cdots,$$ where $1 = \sum_{n=1}^{\infty} a_n \beta^{-n}$
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\( 1_\beta = a_1a_2 \cdots \), where \( 1 = \sum_{n=1}^{\infty} a_n\beta^{-n} \)

**Fact:** \( x \in \Sigma_\beta \iff \sigma^n x \preceq 1_\beta \) for all \( n \)

\( \iff x \) labels a walk starting at B on this graph:

(Here \( 1_\beta = 2100201 \ldots \))
Specification

Topological transitivity \( \Rightarrow \) for every \( w_1, \ldots, w_m \in \mathcal{L} \) \( \exists z_i \in \mathcal{L} \) for which the concatenated word \( w_1 z_1 w_2 z_2 \cdots z_{m-1} w_m \) is in \( \mathcal{L} \).

Definition

\( X \) has **specification** if \( \exists t \in \mathbb{N} \) such that \( z_i \) can always be chosen to have length \( t \), independently of \( w_i \).

- Mixing SFTs and sofic shifts have specification.
- \( \Sigma_{\beta} \) does not have specification if \( 1_{\beta} \) contains arbitrarily long strings of 0’s. (Happens for Leb-a.e. \( \beta > 1 \).)
Specification

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The only obstruction to specification is the tail of the sequence \( 1_\beta \).

**Key idea**: zero entropy obstructions are invisible to MMEs
Obstructions to specification

Let \( \mathcal{G} \subset \mathcal{L} \), where \( \mathcal{L} \) is the set of all words, and let \( \mathcal{G}^M := \{vw \mid v \in \mathcal{G}, |w| \leq M\} \). Define the filtration \( \mathcal{L} = \bigcup_M \mathcal{G}^M \).

**Definition**

\( \mathcal{C} \subset \mathcal{L} \) contains all obstructions to specification if \( \exists \mathcal{G} \) s.t.

- \( \mathcal{L} = \mathcal{G} \mathcal{C} := \{vw \in \mathcal{L} \mid v \in \mathcal{G}, w \in \mathcal{C}\} \)
- Every \( \mathcal{G}^M \) has specification (\( \mathcal{G} \) is a core for specification)

We can glue words (orbit segments) together, provided we are allowed to remove an obstructing piece from the end of each word.
Obstructions to specification

\[ G \subset \mathcal{L} \leadsto G^M := \{vw \mid v \in G, |w| \leq M\} \leadsto \text{filtration } \mathcal{L} = \bigcup_M G^M \]

**Definition**

\( \mathcal{C} \subset \mathcal{L} \) contains all obstructions to specification if \( \exists G \) s.t.

- \( \mathcal{L} = GC := \{vw \in \mathcal{L} \mid v \in G, w \in \mathcal{C}\} \)
- every \( G^M \) has specification (\( G \) is a core for specification)

We can glue words (orbit segments) together, provided we are allowed to remove an obstructing piece from the end of each word.

**Example**

For the \( \beta \)-shift, take \( \mathcal{C} = \{\text{prefixes of } 1_\beta\} \).

- \( \mathcal{C} \) = words whose path never returns to \( B \) (cusp excursions)
- \( G \) = words whose path begins and ends at \( B \)
Small obstructions

**Theorem (C.–Thompson 2010)**

*If $X$ is a shift space, $\mathcal{C}$ contains all obstructions to specification, and $h(\mathcal{C}) < h_{\text{top}}(X, \sigma)$, then $(X, \sigma)$ is intrinsically ergodic.*

**Remark:** If the shadowing orbits can be taken to be periodic, then the unique MME is the limit of the measures $\mu_n = \delta_{\text{Per}(n)}$. 
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**Proposition**

$\pi : X \rightarrow Y$ a factor map, $\mathcal{C} \subset \mathcal{L}(X)$ contains all obstructions $\Rightarrow$

- $\pi(\mathcal{C}) \subset \mathcal{L}(Y)$ also contains all obstructions
- $h(\pi(\mathcal{C})) \leq h(\mathcal{C})$
A shift space $X$ is **coded** if its language $\mathcal{L}$ is freely generated by a countable set of **generators** $\{w_n\}_{n \in \mathbb{N}} \subset \mathcal{L}$.

$$\mathcal{L} = \{\text{all subwords of } w_{n_1} w_{n_2} \cdots w_{n_k} \mid n_i \in \mathbb{N}\}$$

**Canonical decomposition** $\mathcal{L} = C^s G C^p$ s.t. $G(M)$ has specification:

$$G = \{w_{n_1} w_{n_2} \cdots w_{n_k} \mid n_i \in \mathbb{N}\}$$

$$C^s = \{\text{suffixes of } w_n \mid n \in \mathbb{N}\}$$

$$C^p = \{\text{prefixes of } w_n\}$$

**Obstruction:** prefixes and suffixes of generators

Let $\hat{h} = h(\{\text{prefixes and suffixes of generators}\})$.

- $\hat{h} < h_{\text{top}}(X, \sigma) \Rightarrow (X, \sigma)$ is intrinsically ergodic
- $\hat{h} = 0 \Rightarrow$ every subshift factor of $(X, \sigma)$ is intrinsically ergodic
Unique equilibrium states? **Specification is not enough**: consider Manneville–Pomeau map.

- \( f(x) = x + x^{1+\varepsilon} \pmod{1} \), \( \varphi_t = -t \log f' \)
- \( t < 1 \): unique eq. state, fully supported
- \( t > 1 \): unique eq. state, atomic
- \( t = 1 \): two equilibrium states
The Bowen property

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Need control of how \( S_n \varphi \) varies across \( n \)-cylinders.

**Definition**

For \( \mathcal{D} \subset \mathcal{L} \), write \( V_n(\mathcal{D}, \phi) = \sup_{w \in \mathcal{D}_n} \sup_{x, y \in [w]} |\phi(x) - \phi(y)| \). Then \( \varphi \) has the Bowen property on \( \mathcal{D} \) if \( \sup_n V_n(\mathcal{D}, S_n \varphi) < \infty \).

Hölder potentials on shift spaces have the Bowen property.
Utility of the Bowen property

Definition of pressure:

\[ P(D, \varphi) = \lim_{n \to \infty} \frac{1}{n} \log \Lambda_n(D, \varphi) \]

\[ \Lambda_n(D, \varphi) = \sum_{w \in D_n} e^{\sup_{x \in [w]} S_n \varphi(x)} \]

What happens if we replace \( x \) with some other \( y \in [w] \)?

- Continuity \( \Rightarrow \) exponential growth rate \( P(D, \varphi) \) preserved
- Bowen \( \Rightarrow \) \( \Lambda_n \) preserved to within multiplicative constant

Proposition

\( \text{Bowen} + \text{specification} \Rightarrow \frac{1}{C} e^{nP(\varphi)} \leq \Lambda_n(X, \varphi) \leq Ce^{nP(\varphi)} \)
Unique equilibrium states

Theorem ( Bowen 1974)

Consider a system \((X, f)\) and a potential \(\varphi : X \to \mathbb{R}\). Suppose

1. \(X\) a compact metric space;
2. \(f\) a continuous map;
3. \(f\) is expansive;
4. \(f\) has specification;
5. \(\varphi\) has the Bowen property.

Then \(\varphi\) has a unique equilibrium state.

Goal: Replace these with non-uniform versions. Expect to get same result provided obstructions to all properties are small.
Uniqueness in the presence of obstructions

Definition

\( C \subset \mathcal{L} \) contains all obstructions to the Bowen property for \( \varphi \) if

- \( \varphi \) is Bowen on some \( \mathcal{G} \subset \mathcal{L} \) with \( \mathcal{L} = \mathcal{G}C \).

Theorem (C.–Thompson 2011)

Let \( X \) be a shift space and \( \varphi \in C(X) \). If \( C \) contains all obstructions to specification and the Bowen property for \( \varphi \) and \( P(C, \varphi) < P(X, \varphi) \), then there is a unique equilibrium state for \( \varphi \).
Uniqueness in the presence of obstructions

**Definition**

\( \mathcal{C} \subset \mathcal{L} \) contains all obstructions to the Bowen property for \( \varphi \) if

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**Theorem (C.–Thompson 2011)**

Let \( X \) be a shift space and \( \varphi \in C(X) \). If \( \mathcal{C} \) contains all obstructions to specification and the Bowen property for \( \varphi \) and \( P(\mathcal{C}, \varphi) < P(X, \varphi) \), then there is a unique equilibrium state for \( \varphi \).

**Remark:** Each of the following implies that \( P(\mathcal{C}, \varphi) < P(X, \varphi) \).

- \( h(\mathcal{C}) + \sup_{\mu} \int \varphi \, d\mu < P(X, \varphi) \)
- Every equilibrium state \( \mu \) for \( \varphi \) has \( h(\mu) > h(\mathcal{C}) \)
Equilibrium states for $\beta$-shifts

Let $(X, f) = (\Sigma_\beta, \sigma)$ be a $\beta$-shift.

**Theorem (Walters 1978)**

*Every Lipschitz potential $\varphi$ has a unique equilibrium state.*

**Theorem (Hofbauer–Keller 1982)**

*If $\varphi$ has the Bowen property and $\sup \varphi - \inf \varphi < h_{\text{top}}(X, f)$, then $\varphi$ has a unique equilibrium state.*

**Theorem (C.–Thompson 2011)**

*Every Bowen potential $\varphi$ has a unique equilibrium state.*

It suffices to prove that every equilibrium state has positive entropy.
Variants of Manneville–Pomeau

Generalise Manneville–Pomeau:
- $\gamma > 0 \implies f_\gamma(x) = x + \gamma x^{1+\epsilon} \pmod{1}$.
- Coding space is $\Sigma_\beta$ for some $\beta > 1$.
- For most values of $\gamma$, specification fails.
- $\varphi_t = -t \log f'$ is not Bowen.

Remark: For shape of pressure graph, must show
- $\exists \chi > 0$ s.t. $\lambda(x) \geq \chi$ for Leb-a.e. $x$;
- $h(\mu) \leq \lambda(\mu)$ for every $\mu \in M$.

Theorem (C.–Thompson 2011)

For $t < 1$ and $\epsilon \in (0, 1)$, there is a unique equilibrium state for $\varphi_t$. 
Variations of Manneville–Pomeau (cont.)

\( \mathcal{C} \) must capture all obstructions to both specification and the Bowen property.

**Bowen property** ⇔ bounded distortion

1. Bounded distortion on \( \mathcal{D} := \{w \mid w_{|w|} > 0\} \)
2. Take \( \mathcal{C} \) as before and expand it to
   \[ \hat{\mathcal{C}} := \{w0^k \mid w \in \mathcal{C} \cup \sigma \mathcal{C}, k \geq 0\} \]
3. \( \hat{\mathcal{G}} = \{w \in \mathcal{G} \mid w_{|w|} > 0\} \cup \{wb \mid w \in \mathcal{G}\} \)
4. \( P(\mathcal{C}, \varphi_t) = \max(\lim \frac{1}{n} S_n \varphi_t(x), \varphi_t(0)) = \lim \frac{1}{n} S_n \varphi_t(x) \)
5. Uses similar argument to Bowen potentials on \( \Sigma_\beta \) provided \( P(\varphi_t) > 0 \).
Proof sketch for unique MME (Bowen’s proof)

Step 1. Constructible MME $\mu$

Step 2. $\mu$ is ergodic and Gibbs.

Step 3. No room for another MME $\nu \perp \mu$. 
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$$Ke^{-nh} \leq \mu([w]) \leq K'e^{-nh}$$

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Step 0. Counting estimates. $C e^{nh} \leq \#\mathcal{D}_n \leq C' e^{nh}$ whenever $\nu(\mathcal{D}_n) \gg 0$ for some MME $\nu$

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\mu(D_n) \geq Ke^{-nh} \#D_n \geq KC > 0
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Proof sketch for unique MME (Our proof)

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Proof sketch for unique MME (Our proof)

Step 0. Counting estimates. \( C(M) e^{nh} \leq \#(\mathcal{D} \cap \mathcal{G}(M))_n \leq C' e^{nh} \) whenever \( \nu(\mathcal{D}_n) \gg 0 \) for some MME \( \nu \)

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\[
\mu((\mathcal{D} \cap \mathcal{G}(M))_n) \geq K(M)e^{-nh} \#(\mathcal{D} \cap \mathcal{G}(M))_n \geq K(M)C(M) > 0
\]
Bowen potentials on $\beta$-shifts

$X$ a $\beta$-shift and $\varphi$ a Bowen potential. $C$ as before.

- Need to check that $P(C, \varphi) < P(X, \varphi)$.

For a $\beta$-shift, $C_n$ contains exactly one word, so $C$ has zero entropy.

For $\varphi \neq 0$, need to show that $C$ has small pressure:

$$\lim_{n \to \infty} \frac{1}{n} S_n \varphi(1_\beta) < P(X, \varphi).$$

**Key step:** exponentially many words near $1_\beta$
Estimating $P(\mathcal{C}, \varphi)$, Part I

1. Write $1_\beta = u_1 u_2 u_3 \cdots$, where $u_j = 0^{\ell_j} a_j$ for some $a_j > 0$.

2. Can replace any $u_j$ with $\hat{u}_j = 0^{\ell_j} 0$ and get a legal word.

3. Can change any $k$ of the first $n$ words $u_j$ to $\hat{u}_j$; this produces $\binom{n}{k}$ distinct words $w \in \mathcal{L}_N$, where $N = |u_1 u_2 \cdots u_n|$.

4. Each such word has $\inf_{x \in [w]} S_N \varphi(x) \geq S_N \varphi(1_\beta) - 5kV$, where $V = \sup_{w \in \mathcal{L}} \sup_{x, y \in [w]} |S_{|w|} \varphi(x) - S_{|w|} \varphi(y)|$.

5. $\Lambda_N(\mathcal{L}, \varphi) \geq \sum_{k=0}^{n} \binom{n}{k} e^{S_N \varphi(1_\beta) - 5kV} = e^{S_N \varphi(1_\beta)} (1 + e^{-5V})^n$

6. If $1_\beta$ has a positive frequency of non-zero symbols, then $n \geq \delta N$ for some $\delta > 0$, so this suffices.
Estimating $P(\mathcal{C}, \varphi)$, Part II

What if there is not a positive frequency of non-zero symbols?

1. $\frac{n}{N} \to 0 \Rightarrow \delta_{1\beta,n} \xrightarrow{wk^*} \delta_0 \Rightarrow \frac{1}{n} S_n \varphi(1_\beta) \to \varphi(0)$.

2. Fix $1 < \beta' < \beta$ such that $\Sigma_{\beta'}$ is an SFT.

3. $\varphi|_{\Sigma_{\beta'}}$ has a unique equilibrium state, which is not $\delta_0$:

   $\varphi(0) < P(\Sigma_{\beta'}, \varphi) \leq P(\Sigma_\beta, \varphi)$.

If $\lim \frac{n}{N} = 0$ and $\lim \frac{N}{n} > 0$, need to combine both arguments.

Remark: Argument adapts to $f_\gamma$ and $\varphi_t$ provided $P(\varphi_t) > 0$. 
What if there is not a positive frequency of non-zero symbols?

1. \( \frac{n}{N} \to 0 \Rightarrow \delta_{1\beta,n} \xrightarrow{wk^*} \delta_0 \Rightarrow \frac{1}{n} S_n \varphi(1\beta) \to \varphi(0). \)

2. Fix \( 1 < \beta' < \beta \) such that \( \Sigma_{\beta'} \) is an SFT.

3. \( \varphi|\Sigma_{\beta'} \) has a unique equilibrium state, which is not \( \delta_0 \):

\[
\varphi(0) < P(\Sigma_{\beta'}, \varphi) \leq P(\Sigma_{\beta}, \varphi).
\]

If \( \lim \frac{n}{N} = 0 \) and \( \lim \frac{n}{N} > 0 \), need to combine both arguments.

Remark: Argument adapts to \( f_\gamma \) and \( \varphi_t \) provided \( P(\varphi_t) > 0 \).