Motivating examples in dynamical systems

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A pretty picture
The logistic map

**Logistic map:** \( f(x) = \lambda x(1 - x) \quad 0 \leq \lambda \leq 4 \)

- Maps the interval \([0, 1]\) into itself
- Iterate over and over again: represents state of a dynamical system evolving in time

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\( \lambda = 3.5 \)

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More than just a pretty picture

**Bifurcation diagram.** Horizontal $=$ parameter, vertical $=$ recurrent states
More than just a pretty picture

$\lambda \in [3, 3.57 \ldots]$ ← period-doubling cascade
More than just a pretty picture

\[ \lambda \in [3.832, 3.857 \ldots] \leftarrow \text{window of stability} \]
More than just a pretty picture

\[ \lambda = 4 \quad \text{← chaos} \]
Aside: Mandelbrot set

Fix $c \in \mathbb{C}$: let $z_0 = 0$, $z_{n+1} = z_n^2 + c$.

Mandelbrot set $M = \{ c \mid z_n \nrightarrow \infty \}$
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Numerical picture of bifurcation diagram for logistic maps raises questions:

1. Various phenomena are suggested by numerics: period-doubling cascades, windows of stability, self-similarity, chaos. Can their existence be proved rigorously?
2. Qualitative behaviour depends on parameter. How large are the parameter sets on which different behaviours occur?
3. Can consider other one-parameter families of interval maps \( f_\lambda : [0, 1] \). Does the same story happen here?
4. What about higher dimensions \( (f_\lambda : \mathbb{R}^d) \) or manifolds \( (f_\lambda : M) \)?
General answers

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2. Size of parameter sets (prevalence of different behaviours).

3. Other interval maps.

4. Higher dimensions.
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   Numerics suggest a similar story, but proofs are much harder and most answers are still unknown.