Introduction
Intrinsic ergodicity
Unique equilibrium states
Sketch of proofs

Thermodynamics with small obstructions

Vaughn Climenhaga
University of Toronto

October 31, 2011

Joint work with Daniel J. Thompson (Penn State)
1. Introduction

2. Intrinsic ergodicity

3. Unique equilibrium states

4. Sketch of proofs
The talk in one slide

specification $\Rightarrow$ intrinsic ergodicity (unique MME)

Theorem (C.–Thompson)

If all obstructions to specification have small entropy, then the MME is unique.
The talk in one slide

specification $\Rightarrow$ intrinsic ergodicity (unique MME)

Theorem (C.–Thompson)

If all obstructions to specification have small entropy, then the MME is unique.

specification + Bowen property $\Rightarrow$ unique equilibrium state

Theorem (C.–Thompson)

If all obstructions to specification and the Bowen property have small pressure, then the equilibrium state is unique.
Topological pressure

Topological dynamical system:
- $X$ a compact metric space, $f : X \to X$ continuous
- $\mathcal{M} = \{\text{Borel } f\text{-invariant probability measures on } X\}$

Variational principle 1: $h_{\text{top}} (X, f) = \sup_{\mu \in \mathcal{M}} h_\mu (f)$

Variational principle 2: $P(\varphi) = \sup_{\mu \in \mathcal{M}} \left( h_\mu (f) + \int \varphi \, d\mu \right)$

Maximum achieved by MME/equilibrium state

Example

$X = \Sigma_2^+ = \{0, 1\}^\mathbb{N}$, $p, q \in \mathbb{R}$ 

$\varphi(x) = p1[0] + q1[1]$

Then $P(\varphi) = \log(e^p + e^q)$ and the unique equilibrium state is $(\alpha, 1-\alpha)$-Bernoulli, where $

\alpha = e^p / (e^p + e^q)$.
Topological pressure

Topological dynamical system:
- $X$ a compact metric space, $f: X \rightarrow X$ continuous
- $\mathcal{M} = \{\text{Borel } f\text{-invariant probability measures on } X\}$

Variational principle 1: $h_{\text{top}}(X, f) = \sup_{\mu \in \mathcal{M}} h_\mu(f)$

Variational principle 2: $P(\varphi) = \sup_{\mu \in \mathcal{M}} \left( h_\mu(f) + \int \varphi \, d\mu \right)$

Maximum achieved by MME/equilibrium state

Example

$X = \Sigma_2^+ = \{0, 1\}^\mathbb{N}$, $p, q \in \mathbb{R}$, $\varphi(x) = p1_{[0]} + q1_{[1]}$

Then $P(\varphi) = \log(e^p + e^q)$ and the unique equilibrium state is $(\alpha, 1 - \alpha)$-Bernoulli, where $\alpha = \frac{e^p}{e^p + e^q}$.

When is there a unique equilibrium state?
Introduction
Intrinsic ergodicity
Unique equilibrium states
Sketch of proofs

Thermodynamic formalism

Topological pressure $P(\varphi) = \sup_\mu(h_\mu(f) + \int \varphi \, d\mu)$:
- supremum of affine functions $\Rightarrow$ convex function $C(X) \to \mathbb{R}$;
- equilibrium states are tangent functionals in $C(X)^*$.

If entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous, then:
- so is $\mu \mapsto (h_\mu(f) + \int \varphi \, d\mu) \Rightarrow$ existence;
- uniqueness $\Leftrightarrow$ (Gâteaux) differentiability of $P$;
- convex $\Rightarrow$ differentiable on residual set $\Rightarrow$ uniqueness.
Thermodynamic formalism

Topological pressure $P(\varphi) = \sup_\mu (h_\mu(f) + \int \varphi \, d\mu)$:

- supremum of affine functions $\Rightarrow$ convex function $C(X) \rightarrow \mathbb{R}$;
- equilibrium states are tangent functionals in $C(X)^*$.

If entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous, then:

- so is $\mu \mapsto (h_\mu(f) + \int \varphi \, d\mu) \Rightarrow$ existence;
- uniqueness $\Leftrightarrow$ (Gâteaux) differentiability of $P$;
- convex $\Rightarrow$ differentiable on residual set $\Rightarrow$ uniqueness.

This says nothing about which potentials admit unique equilibrium states, or what the ergodic properties of those states are.

**General principle:** Uniform mixing of $X$ and regularity of $\varphi$ should imply uniqueness and strong ergodic properties.
Thermodynamics for shift spaces

Focus on shift spaces (subshifts):

- $X \subset \Sigma^+_p$ closed and $\sigma$-invariant, NOT NECESSARILY AN SFT
- language of $X$: $\mathcal{L} = \mathcal{L}(X) = \{x_1 \cdots x_n \mid x \in X, n \geq 1\}$
- words of length $n$: $\mathcal{L}_n = \{w \in \mathcal{L} \mid |w| = n\}$
- entropy: $h_{top} (X, \sigma) = \lim_{n \to \infty} \frac{1}{n} \log \# \mathcal{L}_n$
- pressure: $P(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{w \in \mathcal{L}_n} e^{\sup_{x \in [w]} S_n \varphi(x)}$
- $S_n \varphi(x) = \varphi(x) + \varphi(\sigma x) + \cdots + \varphi(\sigma^{n-1} x)$

expansive $\Rightarrow$ entropy map $\mathcal{M} \to \mathbb{R}$ upper semi-continuous

Existence guaranteed: the real question is uniqueness.

Beyond transitivity, what do we need?
Uniqueness may fail

Example

Let $X \subset \Sigma_5^+ = \{0, 1, 2, 1, 2\}^\mathbb{N}$ be the shift whose language $\mathcal{L}$ contains $v0^n w$ and $w0^n v$ if and only if $n \geq 2 \max(|v|, |w|)$.

- $(X, \sigma)$ is topologically transitive (indeed, mixing)
- $h_{\text{top}} (X, \sigma) = \log 2$
- 2 measures of maximal entropy:
  
  $\nu = (\frac{1}{2}, \frac{1}{2})$-Bernoulli on $\{1, 2\}^\mathbb{N}$, 
  
  $\mu = (\frac{1}{2}, \frac{1}{2})$-Bernoulli on $\{1, 2\}^\mathbb{N}$.

Uniqueness of an MME can fail for transitive shifts.
Classes of intrinsically ergodic shifts

The following are **intrinsically ergodic** (unique MME):

- Irreducible subshifts of finite type (**Parry 1964**)
Classes of intrinsically ergodic shifts

The following are intrinsically ergodic (unique MME):

- Irreducible subshifts of finite type (Parry 1964)
Classes of intrinsically ergodic shifts

The following are **intrinsically ergodic** (unique MME):

- Irreducible subshifts of finite type *(Parry 1964)*
- Shifts with specification *(Bowen 1974)*
Classes of intrinsically ergodic shifts

The following are intrinsically ergodic (unique MME):

- Irreducible subshifts of finite type (Parry 1964)
- Shifts with specification (Bowen 1974)
- $\beta$-shifts (Walters 1978, Hofbauer 1979)
The motivating question

Intrinsic ergodicity is not necessarily preserved by factors.

- \( X \subset \{0, 1, 2, 1, 2\}^\mathbb{N} \) as before
- \( Y \subset \Sigma_6^+ = \{0, 1, 2, 1, 2, 3\}^\mathbb{N} \) by similar rule
- \( X \) is a factor of \( Y \); \( Y \) is intrinsically ergodic; \( X \) is not

Specification is preserved by factors, so intrinsic ergodicity survives.

What about \( \beta \)-shifts? (Klaus Thomsen, Mike Boyle)
The motivating question

Intrinsic ergodicity is not necessarily preserved by factors.

- $X \subset \{0, 1, 2, 1, 2\}^\mathbb{N}$ as before
- $Y \subset \Sigma_6^+ = \{0, 1, 2, 1, 2, 3\}^\mathbb{N}$ by similar rule
- $X$ is a factor of $Y$; $Y$ is intrinsically ergodic; $X$ is not

Specification is preserved by factors, so intrinsic ergodicity survives.

What about $\beta$-shifts? (Klaus Thomsen, Mike Boyle)

**Theorem (C.–Thompson 2010)**

$(X, \sigma)$ a subshift factor of a $\beta$-shift

$\Rightarrow$ all obstructions to specification have zero entropy

$\Rightarrow (X, \sigma)$ is intrinsically ergodic
For $\beta > 1$, $\Sigma_\beta$ is the coding space for the map

$$f_\beta: [0, 1] \rightarrow [0, 1], \quad x \mapsto \beta x \pmod{1}$$

$1_\beta = a_1 a_2 \cdots$, where $1 = \sum_{n=1}^{\infty} a_n \beta^{-n}$
For $\beta > 1$, $\Sigma_\beta$ is the coding space for the map

$$f_\beta: [0, 1] \rightarrow [0, 1], \quad x \mapsto \beta x \pmod{1}$$

$1_\beta = a_1 a_2 \cdots$, where $1 = \sum_{n=1}^{\infty} a_n \beta^{-n}$

**Fact:** $x \in \Sigma_\beta$ iff $x$ labels a walk starting at $B$ on the graph shown. (Here $1_\beta = 2100201 \ldots$)
Specification

Topological transitivity $\Rightarrow$ for every $w_1, \ldots, w_m \in \mathcal{L}$ $\exists z_i \in \mathcal{L}$ for which the concatenated word $w_1z_1w_2z_2\cdots z_{m-1}w_m$ is in $\mathcal{L}$.

- Mixing SFTs and sofic shifts have specification.
- $\Sigma_\beta$ does not have specification if $1_\beta$ contains arbitrarily long strings of 0’s. (Happens for Leb-a.e. $\beta > 1$.)
Specification

Topological transitivity $\Rightarrow$ for every $w_1, \ldots, w_m \in \mathcal{L}$ $\exists z_i \in \mathcal{L}$ for which the concatenated word $w_1 z_1 w_2 z_2 \cdots z_{m-1} w_m$ is in $\mathcal{L}$.

Definition

$X$ has specification if $\exists t \in \mathbb{N}$ such that $z_i$ can always be chosen to have length $t$, independently of $w_i$.

- Mixing SFTs and sofic shifts have specification.
- $\Sigma_\beta$ does not have specification if $1_{\beta}$ contains arbitrarily long strings of 0’s. (Happens for Leb-a.e. $\beta > 1$.)

The only obstruction to specification is the tail of the sequence $1_{\beta}$.

Key idea: zero entropy obstructions are invisible to MMEs
Obstructions to specification

What do we mean by “obstruction”?  

**Definition**

$\mathcal{G} \subset \mathcal{L}$ is a **core for specification** if

- Every $\mathcal{G}(M) := \{vw \in \mathcal{L} \mid v \in \mathcal{G}, |w| \leq M\}$ has specification

$\mathcal{C} \subset \mathcal{L}$ contains all **obstructions to specification** if $\exists$ core $\mathcal{G}$ s.t.

- $\mathcal{L} = \mathcal{G}\mathcal{C} := \{vw \in \mathcal{L} \mid v \in \mathcal{G}, w \in \mathcal{C}\}$

We can glue words (orbit segments) together, provided we are allowed to remove an obstructing piece from the end of each word.
Obstructions to specification

What do we mean by “obstruction”?

**Definition**

\( G \subset L \) is a **core for specification** if

- Every \( G(M) := \{vw \in L | v \in G, |w| \leq M\} \) has specification.

\( C \subset L \) contains all **obstructions to specification** if \( \exists \) core \( G \) s.t.

- \( L = GC := \{vw \in L | v \in G, w \in C\} \)

We can glue words (orbit segments) together, **provided we are allowed to remove an obstructing piece from the end of each word**.

**Example**

For the \( \beta \)-shift, take \( C = \{\text{prefixes of } 1_\beta\} \).

- \( C \) = words whose path never returns to \( B \) (cusp excursions)
- \( G \) = words whose path begins and ends at \( B \)
Small obstructions

**Theorem (C.–Thompson 2010)**

*If $X$ is a shift space, $C$ contains all obstructions to specification, and $h(C) < h_{\text{top}}(X, \sigma)$, then $(X, \sigma)$ is intrinsically ergodic.*

**Remark:** If the shadowing orbits can be taken to be periodic, then the unique MME is the limit of the measures $\mu_n = \delta_{\text{Per}(n)}$. 
Small obstructions

Theorem (C.–Thompson 2010)

If $X$ is a shift space, $\mathcal{C}$ contains all obstructions to specification, and $h(\mathcal{C}) < h_{\text{top}}(X, \sigma)$, then $(X, \sigma)$ is intrinsically ergodic.

Remark: If the shadowing orbits can be taken to be periodic, then the unique MME is the limit of the measures $\mu_n = \delta_{\text{Per}(n)}$.

Proposition

$\pi : X \rightarrow Y$ a factor map, $\mathcal{C} \subset \mathcal{L}(X)$ contains all obstructions $\Rightarrow$

- $\pi(\mathcal{C}) \subset \mathcal{L}(Y)$ also contains all obstructions
- $h(\pi(\mathcal{C})) \leq h(\mathcal{C})$
Coded systems

A shift space $X$ is **coded** if its language $\mathcal{L}$ is freely generated by a countable set of **generators** $\{w_n\}_{n \in \mathbb{N}} \subset \mathcal{L}$.

$$\mathcal{L} = \{\text{all subwords of } w_{n_1} w_{n_2} \cdots w_{n_k} \mid n_i \in \mathbb{N}\}$$

Canonical decomposition $\mathcal{L} = C^s G C^p$ s.t. $G(M)$ has specification:

$$G = \{w_{n_1} w_{n_2} \cdots w_{n_k} \mid n_i \in \mathbb{N}\}$$

$$C^s = \{\text{suffixes of } w_n \mid n \in \mathbb{N}\}$$

$$C^p = \{\text{prefixes of } w_n\}$$

**Obstruction:** prefixes and suffixes of generators

Let $\hat{h} = h(\{\text{prefixes and suffixes of generators}\})$.

- $\hat{h} < h_{\text{top}}(X, \sigma) \Rightarrow (X, \sigma)$ is intrinsically ergodic
- $\hat{h} = 0 \Rightarrow$ every subshift factor of $(X, \sigma)$ is intrinsically ergodic
The Bowen property

Unique equilibrium states? **Specification is not enough**

**Example (Manneville–Pomeau map)**

- $f(x) = x + x^{1+\varepsilon} \pmod 1$, $\varphi_t(x) = -t \log f'(x)$
- $t \neq 1$: unique equilibrium state, fully supported/atomic
- $t = 1$: two equilibrium states
The Bowen property

Unique equilibrium states? Specification is not enough

Example (Manneville–Pomeau map)

- $f(x) = x + x^{1+\varepsilon} \pmod{1}$, $\varphi_t(x) = -t \log f'(x)$
- $t \neq 1$: unique equilibrium state, fully supported/atomic
- $t = 1$: two equilibrium states

Given $X \subset \Sigma_p^+$ and $\mathcal{D} \subset \mathcal{L}(X)$, the *nth variation* of $\phi$ on $\mathcal{D}$ is

$$V_n(\mathcal{D}, \phi) = \sup_{w \in \mathcal{D}} \sup_{x,y \in [w]} |\phi(x) - \phi(y)|$$

Definition

$\varphi$ has the **Bowen property** if $\sup_n V_n(\mathcal{L}, S_n \varphi) < \infty$. 

Utility of the Bowen property

Definition of pressure:

\[
P(\mathcal{D}, \varphi) = \lim_{n \to \infty} \frac{1}{n} \log \Lambda_n(\mathcal{D}, \varphi)
\]

\[
\Lambda_n(\mathcal{D}, \varphi) = \sum_{w \in \mathcal{D}_n} e^{\sup_{x \in [w]} S_n \varphi(x)}.
\]

What happens if we replace \( x \) with some other \( y \in [w] \)?

- Continuity \( \Rightarrow \) exponential growth rate preserved
- Bowen property \( \Rightarrow \) \( \Lambda_n \) preserved to within multiplicative constant

(\( X \) uniformly expanding \( \Rightarrow \) Hölder continuous functions have the Bowen property.)
Theorem (Bowen 1974)

Consider a system \((X, f)\) and a potential \(\varphi : X \to \mathbb{R}\). Suppose

1. \(X\) a compact metric space;
2. \(f\) a continuous map;
3. \(f\) is expansive;
4. \(f\) has specification;
5. \(\varphi\) has the Bowen property.

Then \(\varphi\) has a unique equilibrium state.

Goal: Replace these with non-uniform versions. Expect to get same result provided obstructions to all properties are small.
Uniqueness in the presence of obstructions

**Definition**

ϕ has the **Bowen property** on G if

- \( \sup_n V_n(G, S_n \varphi) < \infty \)

\( \mathcal{C} \subset \mathcal{L} \) contains all obstructions to the Bowen property for ϕ if

- ϕ is Bowen on some \( G \subset \mathcal{L} \) with \( \mathcal{L} = \mathcal{G} \mathcal{C} \).

**Theorem (C.–Thompson 2011)**

Let X be a shift space and ϕ ∈ C(X). If \( \mathcal{C} \) contains all obstructions to specification and the Bowen property for ϕ and \( P(\mathcal{C}, \varphi) < P(X, \varphi) \), then there is a unique equilibrium state for ϕ.
Uniqueness in the presence of obstructions

Definition

\( \varphi \) has the Bowen property on \( G \) if

\[ \sup_n V_n(G, S_n \varphi) < \infty \]

\( \mathcal{C} \subset \mathcal{L} \) contains all obstructions to the Bowen property for \( \varphi \) if

\( \varphi \) is Bowen on some \( G \subset \mathcal{L} \) with \( \mathcal{L} = GC \).

Theorem (C.–Thompson 2011)

Let \( X \) be a shift space and \( \varphi \in C(X) \). If \( \mathcal{C} \) contains all obstructions to specification and the Bowen property for \( \varphi \) and \( P(\mathcal{C}, \varphi) < P(X, \varphi) \), then there is a unique equilibrium state for \( \varphi \).

Remarks:

- \( \mu_\varphi \) has a weak Gibbs property.
- If \( \mathcal{G}(M) \) has (Per)-specification, then \( \mu_\varphi = \lim_n \delta_{\text{Per}(n)}^\varphi \).
Equilibrium states for $\beta$-shifts

Let $(X, f) = (\Sigma_\beta, \sigma)$ be a $\beta$-shift.

Theorem (Walters 1978)
Every Lipschitz potential $\varphi$ has a unique equilibrium state.

Theorem (Hofbauer–Keller 1982)
If $\varphi$ has the Bowen property and $\sup \varphi - \inf \varphi < h_{\text{top}} (X, f)$, then $\varphi$ has a unique equilibrium state.

Theorem (C.–Thompson 2011)
Every Bowen potential $\varphi$ has a unique equilibrium state.
Variants of Manneville–Pomeau

Example (Generalisation of Manneville–Pomeau)
- $\gamma > 0 \implies f(x) = x + \gamma x^{1+\varepsilon} \mod 1$.
- Topologically (semi-)conjugate to $\Sigma_\beta$ for some $\beta > 1$.
- For most values of $\gamma$, does not have specification.
- $\varphi_t(x) = -t \log f'(x)$ does not have the Bowen property.

Theorem (C.–Thompson 2011)
For $t < 1$ and $\varepsilon \in (0, 1)$, there is a unique equilibrium state for $\varphi_t$. 
Proof sketch for unique MME (Bowen’s proof)

Step 1. Constructible MME \( \mu \)

Step 2. \( \mu \) is ergodic and Gibbs.

Step 3. No room for another MME \( \nu \perp \mu \).
Proof sketch for unique MME (Bowen’s proof)

Step 1. Constructible MME $\mu = \lim_n N(\sum_{w \in \mathcal{L}_n} \delta_{w,n})$

Step 2. $\mu$ is ergodic and Gibbs.

Step 3. No room for another MME $\nu \perp \mu$. 
Proof sketch for unique MME (Bowen’s proof)

Step 1. Constructible MME \( \mu = \lim_n N(\sum_{w \in \mathcal{L}_n} \delta_{w,n}) \)

Step 2. \( \mu \) is ergodic and Gibbs. For \( w \in \mathcal{L}_n \),
\[
K e^{-nh} \leq \mu([w]) \leq K' e^{-nh}
\]

Step 3. No room for another MME \( \nu \perp \mu \).
Proof sketch for unique MME (Bowen’s proof)

Step 1. Constructible MME \( \mu = \lim_n N(\sum_{w \in \mathcal{L}_n} \delta_{w,n}) \)

Step 2. \( \mu \) is ergodic and Gibbs. For \( w \in \mathcal{L}_n \),

\[
K e^{-nh} \leq \mu([w]) \leq K' e^{-nh}
\]

Step 3. No room for another MME \( \nu \bot \mu \). Choose \( D \) such that \( \nu(D_n) \to 1 \) and \( \mu(D_n) \to 0 \).
Proof sketch for unique MME (Bowen’s proof)

Step 0. Counting estimates. \( C e^{nh} \leq \#D_n \leq C' e^{nh} \) whenever \( \nu(D_n) \gg 0 \) for some MME \( \nu \)

Step 1. Constructible MME \( \mu = \lim_n N(\sum_{w \in \mathcal{L}_n} \delta_{w,n}) \)

Step 2. \( \mu \) is ergodic and Gibbs. For \( w \in \mathcal{L}_n \),

\[
Ke^{-nh} \leq \mu([w]) \leq K'e^{-nh}
\]

Step 3. No room for another MME \( \nu \perp \mu \). Choose \( D \) such that \( \nu(D_n) \to 1 \) and \( \mu(D_n) \to 0 \).
Proof sketch for unique MME (Bowen’s proof)

Step 0. Counting estimates. \[ C e^{nh} \leq \#D_n \leq C' e^{nh} \] whenever \( \nu(D_n) \gg 0 \) for some MME \( \nu \)

Step 1. Constructible MME \( \mu = \lim_n N(\sum_{w \in \mathcal{L}_n} \delta_{w,n}) \)

Step 2. \( \mu \) is ergodic and Gibbs. For \( w \in \mathcal{L}_n \),

\[ K e^{-nh} \leq \mu([w]) \leq K' e^{-nh} \]

Step 3. No room for another MME \( \nu \perp \mu \). Choose \( D \) such that \( \nu(D_n) \to 1 \) and \( \mu(D_n) \to 0 \).

\[ \mu(D_n) \geq Ke^{-nh} \#D_n \geq KC > 0 \]
Proof sketch for unique MME (Our proof)

Step 0. Counting estimates. $C e^{nh} \leq \# \mathcal{D}_n \leq C' e^{nh}$ whenever $\nu(\mathcal{D}_n) \gg 0$ for some MME $\nu$

Step 1. Constructible MME $\mu = \lim_n N(\sum_{w \in \mathcal{L}_n} \delta_{w,n})$

Step 2. $\mu$ is ergodic and Gibbs. For $w \in \mathcal{L}_n$,

$$K e^{-nh} \leq \mu([w]) \leq K' e^{-nh}$$

Step 3. No room for another MME $\nu \perp \mu$. Choose $\mathcal{D}$ such that $\nu(\mathcal{D}_n) \to 1$ and $\mu(\mathcal{D}_n) \to 0$.

$$\mu(\mathcal{D}_n) \geq K e^{-nh} \# \mathcal{D}_n \geq KC > 0$$
Proof sketch for unique MME (Our proof)

Step 0. Counting estimates. \( C(M) e^{nh} \leq \#(D \cap G(M))_n \leq C'e^{nh} \) whenever \( \nu(D_n) \gg 0 \) for some MME \( \nu \)

Step 1. Constructible MME \( \mu = \lim_n N(\sum_{w \in \mathcal{L}_n} \delta_{w,n}) \)

Step 2. \( \mu \) is ergodic and Gibbs. For \( w \in \mathcal{L}_n \),

\[
Ke^{-nh} \leq \mu([w]) \leq K'e^{-nh}
\]

Step 3. No room for another MME \( \nu \perp \mu \). Choose \( D \) such that \( \nu(D_n) \to 1 \) and \( \mu(D_n) \to 0 \).

\[
\mu(D_n) \geq Ke^{-nh} \#D_n \geq KC > 0
\]
Proof sketch for unique MME (Our proof)

Step 0. Counting estimates. \( C(M) e^{nh} \leq \#(D \cap \mathcal{G}(M))_n \leq C' e^{nh} \)
whenever \( \nu(D_n) \gg 0 \) for some MME \( \nu \)

Step 1. Constructible MME \( \mu = \lim_n N(\sum_{w \in \mathcal{L}_n} \delta_{w,n}) \)

Step 2. \( \mu \) is ergodic and Gibbs. For \( w \in \mathcal{G}(M)_n \),
\[
K(M) e^{-nh} \leq \mu([w]) \leq K' e^{-nh}
\]

Step 3. No room for another MME \( \nu \perp \mu \). Choose \( D \) such that
\( \nu(D_n) \to 1 \) and \( \mu(D_n) \to 0 \).
\[
\mu(D_n) \geq K e^{-nh} \#D_n \geq KC > 0
\]
Proof sketch for unique MME (Our proof)

Step 0. Counting estimates. $C(M)e^{nh} \leq \#(\mathcal{D} \cap \mathcal{G}(M))_n \leq C'e^{nh}$ whenever $\nu(\mathcal{D}_n) \gg 0$ for some MME $\nu$

Step 1. Constructible MME $\mu = \lim_n N(\sum_{w \in \mathcal{L}_n} \delta_{w,n})$

Step 2. $\mu$ is ergodic and Gibbs. For $w \in \mathcal{G}(M)_n$,

$$K(M)e^{-nh} \leq \mu([w]) \leq K'e^{-nh}$$

Step 3. No room for another MME $\nu \perp \mu$. Choose $\mathcal{D}$ such that $\nu(\mathcal{D}_n) \to 1$ and $\mu(\mathcal{D}_n) \to 0$.

$$\mu((\mathcal{D} \cap \mathcal{G}(M))_n) \geq K(M)e^{-nh} \#(\mathcal{D} \cap \mathcal{G}(M))_n \geq K(M)C(M) > 0$$
Bowen potentials on $\beta$-shifts

In general, choose $\mathcal{G}$ to deal with two sources of bad behaviour:

1. failure of specification;
2. failure of the Bowen property.

$X$ a $\beta$-shift and $\varphi$ a Bowen potential. Decomposition as before.

- Need to check that $P(C, \varphi) < P(X, \varphi)$. 

For a $\beta$-shift, $C_n$ contains exactly one word. $\varphi = 0 \Rightarrow P(C, \varphi) = h(C) = 0 < h(L) = P(X, \varphi)$.

For $\varphi \neq 0$, need a new argument that $P(C, \varphi) < P(X, \varphi)$. Amounts to checking $\lim_{n \to \infty} \frac{1}{n} S_n \varphi(1_\beta) < P(X, \varphi)$. 

Bowen potentials on $\beta$-shifts

In general, choose $G$ to deal with two sources of bad behaviour:

1. failure of specification;
2. failure of the Bowen property.

$X$ a $\beta$-shift and $\varphi$ a Bowen potential. Decomposition as before.

- Need to check that $P(C, \varphi) < P(X, \varphi)$.

For a $\beta$-shift, $C_n$ contains exactly one word.

$$\varphi = 0 \implies P(C, \varphi) = h(C) = 0 < h(L) = P(X, \varphi)$$

For $\varphi \neq 0$, need a new argument that $P(C, \varphi) < P(X, \varphi)$.

Amounts to checking

$$\lim_{n \to \infty} \frac{1}{n} S_n \varphi(1_\beta) < P(X, \varphi).$$
Estimating $P(\mathcal{C}, \varphi)$, Part I

- Write $1_\beta = u_1 u_2 u_3 \cdots$, where $u_j = 0^{\ell_j} a_j$ for some $a_j > 0$.
- Can replace any $u_j$ with $\hat{u}_j = 0^{\ell_j} 0$ and get a legal word.
- Let $N = \sum_{j=1}^{n} (\ell_j + 1) = |u_1 u_2 \cdots u_n|$. Changing $k$ of the first $n$ words $u_j$ to $\hat{u}_j$ gives $\binom{n}{k}$ distinct words $w \in \mathcal{L}_N$.
- For each $w$, every $x \in [w]$ has $S_N \varphi(x) \geq S_N \varphi(1_\beta) - 5kV$, where $V = \sup_m \sup_{w \in \mathcal{L}_m} \sup_{x,y \in [w]} |S_m \varphi(x) - S_m \varphi(y)|$.
- $\Lambda_N(\mathcal{L}, \varphi) \geq \sum_{k=0}^{n} \binom{n}{k} e^{S_N \varphi(1_\beta) - 5kV} = e^{S_N \varphi(1_\beta)} (1 + e^{-5V})^n$.
- If $1_\beta$ has a positive frequency of non-zero symbols, then $n \geq \delta N$ for some $\delta > 0$, so this suffices.
If \( \frac{n}{N} \to 0 \), then \( \delta_{1, n} \to \delta_0 \) in weak* topology, so
\[
\frac{1}{n} S_n \varphi(1_\beta) \to \varphi(0).
\]

Fix \( 1 < \beta' < \beta \) such that \( \Sigma_{\beta'} \) is an SFT.

Then there is a unique equilibrium state for \( \varphi|_{\Sigma_{\beta'}} \), which is fully supported, so
\[
\varphi(0) < P(\Sigma_{\beta'}, \varphi) \leq P(\Sigma_{\beta}, \varphi).
\]

If \( \lim \frac{n}{N} = 0 \) and \( \lim \frac{n}{N} > 0 \), need to combine both arguments.