Broadly, a dynamical system is a set \(X\) with a map \(f : X \to X\). This is discrete time. Continuous time considers a flow \(\varphi_t : X \to X\). We will mostly consider discrete time.

Often \(X\) has some extra structure that the map \(f\) respects.

- \(X\) a smooth manifold, \(f\) a diffeomorphism
- \(X\) a metric space, \(f\) continuous
- \((X, \mu)\) a measure space, \(f\) measure-preserving

\(f\) is measure-preserving / \(\mu\) is \(f\)-invariant: \(\mu(f^{-1}E) = \mu(E)\) for all measurable \(E \subseteq X\). Equivalently, \(\int \varphi \circ f \, d\mu = \int \varphi \, d\mu\) for all \(\varphi \in L^1\).

Classical source of examples: \(X\) is a smooth manifold, \(\varphi_t\) is the flow of a conservative vector field. Then each \(\varphi_t\) both respects smooth structure and preserves volume.

Smooth manifolds have many measures, not just volume. But having an invariant measure opens up the rich toolbox of ergodic theory. For example, “time average = space average” (Birkhoff ergodic theorem).

Aside: What about the dissipative case? What measure should we use instead of volume, when volume is not invariant? Big question, skip for now.

Connections between topological and measure-theoretic structure are illustrated by two “toy” examples on \(X = S^1 \subseteq \mathbb{C}\).

1. \(R_\alpha : z \mapsto z e^{2\pi i \alpha}\) for \(\alpha\) an irrational parameter.
2. \(T_2 : z \mapsto z^2\).

These represent two extremes of dynamical behaviour: \(R_\alpha\) is elliptic, \(T_2\) is hyperbolic.

Date: February 25, 2013.
First consider these topologically. Both are **topologically transitive** – any two open sets can be connected by an orbit. This is an irreducibility criterion.

*Aside:* Transitivity equivalent to existence of a dense orbit. Weaker than **minimality** – every orbit is dense. $R_\alpha$ is minimal, $T_2$ is not.

What about invariant measures? For both, Lebesgue measure is invariant and **ergodic**: every $f$-invariant set $E$ has $\mu(E) = 0$ or 1.

This implies, via **Birkhoff ergodic theorem**: if $\varphi \in L^1$, then for Leb-a.e. $x$,

$$\frac{1}{n} S_n \varphi(x) = \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k x) \to \int \varphi \, dx.$$

This is the **law of large numbers** for the “random variables” $\varphi, \varphi \circ f, \varphi \circ f^2, \ldots$” What about other statistical properties, and the nature of this convergence?

- Is this convergence uniform in $x$?
- How quickly does convergence happen? Look at $E_n := \{ x \mid \frac{1}{n} S_n \varphi(x) > \epsilon \}$. How quickly does the measure of $E_n$ go to 0?

**Fact:** Although ergodicity of Lebesgue measure determines the asymptotic behaviour of Lebesgue-a.e. trajectory for both $R_\alpha$ and $T_2$, the nature of the convergence to this asymptotic behaviour is strongly contingent on the presence of other invariant measures.

$R_\alpha$: Lebesgue is the **only** invariant measure.

$T_2$: There are many, many others. Any periodic orbit supports an invariant (ergodic) measure, and there are $2^n$ fixed points of $T_2^n$.

Given $f: X \to X$, let $\mathcal{M}_f$ be the collection of $f$-invariant Borel probability measures on $X$, and $\mathcal{M}_f^e$ the set of ergodic measures.

Geometrical interpretation: $\mathcal{M}_f^e$ is the set of extreme points of $\mathcal{M}_f$, and $\mathcal{M}_f$ is a **simplex** – elements of $\mathcal{M}_f$ are in 1-1 correspondence with probability measures on $\mathcal{M}_f^e$ (**ergodic decomposition**).

Consider $R_\alpha$ on $k$ concentric circles. Each circle has exactly one ergodic measure. $\mathcal{M}_f$ is a $(k-1)$-simplex.

**Question:** When do two systems have the same $\mathcal{M}_f$ and $\mathcal{M}_f^e$? (Up to affine homeomorphism.)
First invariant: Number of extreme points (ergodic measures). Finite-dimensional simplices are affinely homeomorphic iff same number of extreme points. Also distinguishes countable/uncountable.

Consider $R_\alpha$ on countably many concentric circles, and $R_\alpha$ on unit disc. First has countable $\mathcal{M}_f^c$, second has uncountable.

Second invariant: Topology of $\mathcal{M}_f^c$. Becomes important when $\mathcal{M}_f^c$ infinite. All examples of $R_\alpha$ have $\mathcal{M}_f^c$ closed, while $T_2$ has $\mathcal{M}_f^c$ dense in $\mathcal{M}_f$.

Last property is important. Simplex with dense extreme points constructed in 1961 by E Poulsen. Abstract construction, no dynamics.

Universality of Poulsen simplex: In 1978, J Lindenstrauss, G Olsen, Y Sternfeld showed that if two simplices both have dense extreme points then they are affinely homeomorphic.

The extreme set of Poulsen’s simplex is path-connected. So two conclusions from fact that (countable) set of periodic measures is dense in $\mathcal{M}_f$ for $T_2$:

- existence of uncountably many other ergodic measures;
- path-connectedness of $\mathcal{M}_f^c$.

Questions: How to describe other ergodic measures concretely? For which other systems is $\mathcal{M}_f$ the Poulsen simplex? What is connection between this fact and statistical properties?

Aside: Natural to ask for example of system where $\mathcal{M}_f^c$ is path-connected but not dense. $R_\alpha$ on disc does it but in a silly way - disjoint union of closed subsystems, and $\mathcal{M}_f^c$ is only one-dimensional.

A more sophisticated example is the Dyck shift. $X \subset \{0,1,2,3\}^\mathbb{Z}$ defined by syntax rules on brackets, identifying 0,1,2,3 with (,),[]. Map $f$ is the left shift. Can show $\mathcal{M}_f^c$ connected but not dense.

Return to questions. Useful to think of other symbolic systems where $X \in \Sigma_2^+$ defined by $\Sigma_2^+ := \{0,1\}^\mathbb{N}$ and $f = \sigma$. Connect to maps such as $T_2$ by fixing a partition of $S^1$ into two subsets and labelling each subset with 0 or 1.

For $T_2$, get $X = \Sigma_2^+$. Measure $\mu$ defined by $\mu[w]$, where $w \in \{0,1\}^*$ and $[w]$ is set of sequences starting with $w$. Two important classes:

- $p_1 + p_2 = 1 \Rightarrow$ Bernoulli measure $\mu[w] = p_{w_1} \cdots p_{w_n}$. 
• stochastic $2 \times 2$ matrix $\Rightarrow$ Markov $\mu[w] = p_{w_1} P_{w_1 w_2} \cdots P_{w_{n-1} w_n}$, where $p$ a left eigenvector for $P$.

For $T_2$, no restrictions on what symbol sequences can appear. Corresponds to configurations on lattice: each site can be on or off, + or -, $\uparrow$ or $\downarrow$. Suggests language of statistical mechanics.

Can code $R_\alpha$ by $X \subset \Sigma_2$. Many restrictions, some very long-range.

Interactions of uniformly bounded range: subshift of finite type. More generally, specification property.

• Transitivity for shift space $X$ means any set of words can be concatenated by putting some “buffers” in between.
• Specification means the buffers are uniformly short.

In 1970, K Sigmund showed that specification implies $\mathcal{M}_f^\epsilon$ is dense, hence $\mathcal{M}_f$ is the Poulsen simplex.

The space of invariant measures is often very large – how do we select a distinguished measure?

Topological entropy: exponential growth rate of number of words of length $n$. Call it $h(X)$.

Measure-theoretic entropy: growth rate of number of words of length $n$ needed to get to mass $\frac{1}{2}$. Call it $h(\mu)$.

Variational principle: $h(X) = \sup \{ h(\mu) \mid \mu \in \mathcal{M}_f^\epsilon \}$.

Pressure: Give words weights according to a potential function $\varphi \in C(X)$. Still get variational principle. Measure achieving supremum is an equilibrium state.

Aside: For smooth systems, another notion of distinguished measure is SRB measure. I have active research on these.

Various properties of $\mathcal{M}_f$ and $\mathcal{M}_f^\epsilon$:

• (C) $\mathcal{M}_f^\epsilon$ is path-connected.
• (D) $\mathcal{M}_f^\epsilon$ is dense in $\mathcal{M}_f$.
• (H) $\mathcal{M}_f^\epsilon$ is entropy-dense in $\mathcal{M}_f$ – can approximate in weak* and in entropy.
• (E) There exists a dense subspace $V \subset C(X)$ such that each $\varphi \in V$ has a unique equilibrium state.

SFT $\Rightarrow$ specification $\Rightarrow$ (E), (H), (D)
Conjecture: (E) implies (H). *(The idea is that (E) gives a way to map a very large vector space homeomorphically into \( \mathcal{M}_µ \). The image should be “large enough”).*

(E) implies various multifractal results. (VC, Nonlinearity)

(H) and (E) are important for large deviations properties: recall sets 
\[ E_n = \{ x \mid \frac{1}{n} S_n \varphi(x) > \epsilon \}, \text{ where } \int \varphi \, dx = 0. \]

\[
\lim_{n \to \infty} \frac{1}{n} \text{Leb}(E_n) = \sup \{ h(\mu) - \log 2 \int \varphi \, d\mu > \epsilon \}.
\]

Can get similar results anytime (E) holds (H Comman, J Rivera–Letelier 2010).

Problem: Specification is a very uniform phenomenon, and hence somehow rare. What non-uniform versions still give (E), LDP, etc?

*Example:* Fix \( \beta > 1 \), let \( T_\beta : x \mapsto \beta x \) (mod 1). Code this into \( X_\beta \subset \Sigma_b^+ \), where \( b = [\beta] \). Typically specification fails. But \( X_\beta \) has (E). (VC, DJ Thompson, 2013) Can use this to get LDP. (VC, DJ Thompson, K Yamamoto, in progress)