**Specification & towers (Maryland)**

**Setting**

\[ X \text{ nonempty, } f : X \to \text{cts, } \]

\[ M_f = \text{Borel f-invariant prob} \]

\[ \psi : X \to \mathbb{R} \text{ cts, } \]

or \[ m \in M \text{ a of meas} \]

**Search for**

- equilibrium state (maximises \( h_\mu (f) + \text{Sd}_\mu \))
- or a.c.i.p. \( (\mu \in M_f, \mu \ll m) \) (q, SRB)

**Questions?**

1. Existence?
2. Uniqueness?
3. Properties?

For now let \( X \) be a shift space, \( \psi \) Hölder

**Fact** \( X \) an SFT \( \Rightarrow \exists ! \mu \in M \), \( \mu = \mu \psi \), and \( \mu \) has

**[EDC]** exponential decay of correlations:

\[ \psi, \psi_2 \in C^\alpha \Rightarrow |S_{\psi, \psi_2} \psi_2 \psi_2 d\mu - S_{\psi_2} \psi_2 \psi_2 \psi_2 d\mu| \leq Ce^{-\delta n} \]

**[CLT]** central limit theorem:

\[ \psi \in C^\alpha, \ S_{\psi} \psi_2 = 0 \Rightarrow \sqrt{n} S_n \psi \to \text{Normal in distribution} \]

Smooth systems - same holds for mixing Axiom A. (unif hyp)

What is special about SFTs? (or Axiom A?)

1. Markov structure gives good control of Rep (RPF operator) → largest gap results from functional analysis
2. Specification property → uniform mixing

* 1 gives \( \exists ! \), EDC + CLT, 2 only gives \( \exists ! \)
Given \( X \subset A^\mathbb{N} \) closed, \( \sigma \)-invariant, let \( L = \text{language of } X \) specification = \( \exists \tau \in \mathbb{N} \) s.t. \( \forall u, v \in L \exists w \in L \) with \( w \upharpoonright \tau = \tau \) and \( u w v \in L \).

For non-uniformly hyperbolic systems, get \( \exists + \sqrt{D} \text{EDC/CLT} \) using inducing schemes / Young towers
* embed full shift on \( A^\mathbb{N} \) alphabet into system

Can specification be similarly extended?
used to get \( EDC + CLT \)?

Toy example: \( S \)-gap shifts
- Fix \( S \subset \mathbb{N} \), define \( X \subset \mathbb{S} \) by allowing
  \( 10^n 1 \) iff \( n \in S \).
- \( SPT \leftrightarrow S \) finite
- \( S \text{ periodic } \leftrightarrow S \text{ eventually periodic } \)
- \( S \text{ lacunary } \leftrightarrow S \text{ non-lacunary (bold gaps) } \)
- Represent by \( \mathbb{I} \mathbb{I} \mathbb{I} \) graph:
  \( \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \)
  returns only at elements of \( S \).

\( \text{Rank} \ \{ 0^n 1 \ | \ n \in S \} \) gives a tower

Say \( 0 = L \) has spec if satisfies (*) above:
\( \exists \tau \exists u, v \in L \exists w \in L \) s.t. \( w \upharpoonright \tau = \tau \) \& \( u w v \in L \).
$Y = 50 \qquad n \in S^2 \text{ has } \underbrace{\Theta}_1 = \overbrace{\Theta} w/ \tau = 0$.

Need $Y$ to be "large enough".

1. There exists $C^0, C^5 \leq L$ such that $L \leq C^0 \& C^5$ and $h(C^0 \cup C^5) < h(L) = h_{\text{top}}(X, \tau)$

Here $h(D) = \lim_n \frac{1}{n} \log \# D_n$

(can generalize everything to $\theta \neq 0$, $P(C^0)$)

Write $Y^M = \{uvw \mid u \in C^0, \quad v \in Y, \quad w \in C^5, \quad \|u\|, \|w\| \leq M\}$

**Thm (C., - Thompson 2012)**

If $1$ holds $\& Y^M \& 2$ holds for $Y$, then $(X, \sigma)$ has unique MME. (ES w/ appropriate mods)

- Does this unique MME have EDC, CLT?
- What if $\tau$ has spec?

Observation: if $1 + 2$, then tower with exponential tails $\Rightarrow$ EDC + CLT.

**Rmk:** Spec seems easier to check in smooth setting than a tower does. Also $1 (2)$ go to factors, while towers do not.
A. Bertrand, 1988: If \( X \) has Spec then \( \exists s \in L \) s.t. \( s \cdot v, s \cdot w \in L \implies s \cdot v \cdot w \in L \) (synchronizing).

- non-symbolic interpretation: expansive + spec \( \implies \) local product structure at some point.

Let \( B = \{ v \mid s \cdot v \cdot s \cdot v \in L \} \) (bridging words)

\[ F = s \cdot B = \{ s \cdot v \mid \forall v \in B \} \]

Then \( F \) has \( \mathbb{E} \)-free concatenations (\( \vdash \text{tower + EDC + CLT} \)).

1. To address non-uniform case need small extra condition

   Either of the following holds
   (a) \( \exists v \in X \) s.t. \( v \cdot \chi \cap X < C \cdot \chi \) (or \( v \cdot \chi \cap X < C \cdot \chi \))
   (b) \( v \cdot \chi, w \cdot \chi, v \cdot \chi \in B, k \cdot \chi \rightarrow v \cdot \chi \)

Both are natural for some smooth examples:
   (a) PH systems (C. Fisher-Thompson)
   (b) effective hyperbolicity (C. - Pardini)

1. Thm (C., 2014)
   - If \( \exists s \) has \( \mathbb{E} \) then \( \exists r \in \mathbb{E}, c \in L \) s.t. \( s = r \cdot c \cdot r \in \mathbb{E} \),
     \( B = \{ v \cdot w \cdot \chi \mid r \cdot w \cdot \chi \in \mathbb{E} \} \)
   - If \( \exists s \) has \( \mathbb{I} \) \& \( \mathbb{Y} \), so does \( F : \exists \mathbb{Z} \cdot \mathbb{E} \cdot \mathbb{Z} < h(\mathbb{E} \cup \mathbb{Z}) \).

1. Cor \( \mathbb{Y} \rightarrow \exists ! \mu \text{ MME, } (X, \mu) \text{ has tower with exp tails, } \mu \text{ has EDC \& CLT} \)