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Math 3333 - Intermediate Analysis - David Blecher
KEY—Test 2–July 29, 2010.

Instructions. Time= 1 hour. Show all working and reasoning, the points are almost all for logical, complete reasoning. You may quote without proofs results from the classnotes or text, except for the part you are asked to prove. [Approximate point values are given, total = 100 points plus 9 bonus points].

1. (a) Show that a convergent sequence is bounded. [12]
   (b) State the Bolzano-Weierstrass theorem for sequences. [4]
   (c) Prove that if $s_n \to s$ then $|s_n| \to |s|$. [6]
   (d) Give the definition (involving ϵ) of a Cauchy sequence. [5]
   (e) State the Cauchy test. [4]

   Solution: (a) Suppose $s_n \to s$, then by ~~(a)~~ definition with $\epsilon = 1$ there exists N such that $|s_n - s| < 1$ whenever $n \geq N$. Now $|s_n| = |s_n - s + s| \leq |s_n - s| + |s| < 1 + |s|$ whenever $n \geq N$. Let $M = \max\{|s_1|, |s_2|, \cdots, |s_N|, 1 + |s|\}$. then clearly $|s_n| \leq M$ if either $n < N$ or $n \geq N$.

   (b) Every bounded sequence has a convergent subsequence.

   (c) Method 1: $|s_n - s| \to 0$ by 'Fact 2', so $||s_n| - |s|| \leq |s_n - s| \to 0$. By Fact 6, $|s_n| \to |s|$.

   Method 2: The function $f(x) = |x|$ is continuous, so by the 'main theorem for continuity': $f(s_n) \to f(s)$.

   (d) That given any $\epsilon > 0$, $\exists N$ such that $|s_n - s_m| < \epsilon$ whenever $m \geq n \geq N$.

   (e) A sequence is convergent iff it is a Cauchy sequence.

2. Prove that $\lim_n \frac{\sin(n)}{2n+1} = 0$. [5]

   Solution: $\left|\frac{\sin(n)}{2n+1}\right| \leq \frac{1}{2n} \to 0$. Therefore by 'Fact 6', $\lim_n \frac{\sin(n)}{2n+1} = 0$.

3. Complete the sentence: A set S is closed iff for every sequence (s_n) in S, if (s_n) converges then ______________________. [3]

   Solution: ... its limit is in S too.

4. Using the ϵ-δ definition, show that $\lim_{x \to 1} \frac{2x^2+3x+3}{x+1} = 4$. [17]

   Solution: $\frac{2x^2+3x+3}{x+1} - 4 = \frac{2x^2-x-1}{x+1} = \frac{(2x+1)(x-1)}{x+1}$. Thus if $|x-1| < 1$ then $0 < x < 2$ and so $x+1 > 1$. Also, $0 < 2x < 4$ and so $1 \leq 2x+1 \leq 5$. Therefore

$$\left|\frac{2x^2+3x+3}{x+1} - 4\right| = \frac{|2x+1||x-1|}{|x+1|} < 5|x-1|.$$

Given $\epsilon > 0$ choose $\delta = \min\{1, \epsilon/5\}$. Then $0 < |x-1| < \delta$ implies that $\left|\frac{2x^2+3x+3}{x+1} - 4\right| < 5|x-1| < 5\delta \leq \epsilon$.

if $\delta < 1$

Alternatively: $\frac{|2x+1||x-1|}{|x+1|} \leq \frac{|2(x-1)+3||x-1|}{|(x-1)+2|} \leq \frac{(2|x-1|+3)|x-1|}{2-|x-1|} < \frac{(2\delta+3)\delta}{2-\delta} < \frac{5\delta}{1}$,

5. Suppose that $f : (a, b) \to \mathbb{R}$, $g : (a, b) \to \mathbb{R}$, $L \in \mathbb{R}$, and $a < c < b$.

   (a) Define what it means for f to be continuous at c (in terms of ϵ-δ). [5]

   (b) Prove that if f is continuous at c then $f(s_n) \to f(c)$ for every sequence (s_n) converging to c. [7]

   (c) Prove that if $f(x)$ and $g(x)$ are continuous at c, then $f(x) + g(x)$ is continuous at c. [8]

   (d) Prove that if $f(x)$ is continuous at c, and $f(c) > 0$, then there is a neighborhood U of c such that $f(x) > f(c)/2$ for all $x \in U$. [8]

Soln. (a) For every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$.

(a)

(b) Given $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. Since $s_n \to c$ there exists N such that $|s_n - c| < \delta$ if $n \geq N$. Thus if $n \geq N$ then $|f(s_n) - f(c)| < \epsilon$. That is, $f(s_n) \to f(c)$.

(c) If $s_n \to c$, then $f(s_n) \to f(c)$ and $g(s_n) \to g(c)$ (by (b)). By Fact 9 for sequences, $f(s_n) + g(s_n) \to f(c) + g(c)$. By the converse of (b) (Main Theorem # 2), $f(x) + g(x)$ is continuous at c.

(d) Let $\epsilon = f(c)/2$, then by definition, there is a $\delta > 0$ such that $|f(x) - L| < \epsilon = L/2$ when $|x - c| < \delta$. Now $|f(x) - L| < \epsilon = L/2$ implies that $f(x) > L - L/2 = L/2$. Thus on the neighborhood $U = N(c, \delta)$ we have $f(x) > L/2$.

6. (a) Complete the sentence: A nonempty set S is compact iff every sequence in S has a ________________________ [4]

   (b) State and prove the Min-Max/Extreme Value Theorem. Your proof should contain the proof about Range(f) being compact. [21]

Soln. (a) ... subsequence that converges to a number in S.

or $[a,b]$

(b) The 'min-max theorem' states that if C is a compact set and if $f : C \to \mathbb{R}$ is continuous, then there exist $x_1, x_2 \in C$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in C$. [4]

We first use the criterion in (a) to show that $f(C)$ is compact. So let (y_n) be a sequence in $f(C)$. If $y_n = f(x_n)$ then (x_n) is a sequence in C, so by (a), there is a convergent subsequence $x_{n_k} \to x \in C$. By the Main Theorem # 2, $y_{n_k} = f(x_{n_k}) \to f(x)$, and $f(x) \in f(C)$. So by the criterion in (a), $f(C)$ is compact.

Any compact set in $\mathbb{R}$ has a minimum and a maximum element, by another theorem from class. Thus there exist $x_1, x_2 \in C$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in C$. ~~[6 ADD]~~

(In class we also said in the Min-Max Theorem that $f(C)$ was compact.)