Test 2 - Advanced Multivariable Calculus

Instructions: Time = 1 1/2 hour. You may use freely any theorems you can quote from the class notes (except of course those you are asked to prove).

Q1.

(i) If $\mathcal{O}$ is an open set in $\mathbb{R}^k$, and if $f : \mathcal{O} \to \mathbb{R}$, define the notions "directional derivative of f", "total derivative of f" and "$f \in C^k(\mathcal{O})$".

(ii) State what it means for f to be differentiable, and show how this implies the 'tangent plane (or first order linear) approximation.

(iii) Prove that if $f \in C^1(\mathcal{O})$ and if $\underline{v}$ is a unit vector, then $(D_{\underline{v}}f)(\underline{p}) = \underline{v} \cdot f'(\underline{p})$ for any $\underline{p} \in \mathcal{O}$.

Q2. Consider a change of coordinates transformation given by the following formula:

$$\begin{cases} u = x^2 - y^2 \\ v = xy \end{cases}$$

(i) Write the transformation as a single function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$. What is D, n and m?

(ii) Find a function of the form $T(\overrightarrow{x}) = A\overrightarrow{x} + \overrightarrow{b}$ which approximates the transformation nearby the point $(2, -1)$. Here A and $\overrightarrow{b}$ are matrices or vectors with constant entries.

(iii) Approximate u and v when $(x, y) = (1.9, -1.2)$.

Q3. State and prove the first and second derivative tests.

Q4. In the following $f : [a, b] \times [c, d] \to \mathbb{R}$ is bounded.

(i) Define the terms 'partition (or dicing) of $[a, b] \times [c, d]$', 'mesh', Riemann sum.

(ii) Give four equivalent definitions of f being Riemann integrable.

(iii) Prove that if f and g are Riemann integrable on the square $D = [a, b] \times [c, d]$ then $f + g$ is Riemann integrable on D and $\int_D (f + g)\, dxdy = \int_D f\, dxdy + \int_D g\, dxdy$.

(iv) Prove that if f is continuous then it is integrable.

Q5. State the list of 9 properties of the integral on a subset of $\mathbb{R}^n$.

Q6. Consider the system

$$\begin{cases} u = x + xyz \\ v = y + xy \\ w = z + 2x + 3z^2. \end{cases}$$

(i) Show using the inverse function theorem that it is possible to solve for x, y, and z above explicitly in terms of u, v, w, in a neighborhood of the point $(x, y, z) = (0, 0, 0)$.

(ii) Using the inverse function theorem find the Jacobian matrix of derivatives of x, y, and z with respect to u, v, w, at the point where $(x, y, z) = (0, 0, 0)$.

(iii) If R is a small box in uvw-space with one corner at $(0, 0, 0)$, show how to approximate the volume of the corresponding 'deformed box' in the $xyz-$space.