Solutions: HW 1 (Chapter 2)

1. Let X be the value of the contestant's accumulated winnings in thousands of dollars after the spin. Note that the value of the contestant's accumulated winnings before the spin is $15,000.

   a. The probability mass function is given by

   $p_X[0] = \frac{1}{12},\ p_X[15.5] = \frac{1}{2},\ p_X[16] = \frac{1}{4},\ p_X[17] = \frac{1}{12},\ p_X[20] = \frac{1}{12},$

   $p_X[x] = 0$ for all other x.

   b. The graph of p_X is given by

   (Omitted)

c. The expected value of X is given by

$$E[X] = \sum x\, p_X[x] = (0)\left(\frac{1}{12}\right) + (15.5)\left(\frac{1}{2}\right) + (16)\left(\frac{1}{4}\right) + (17)\left(\frac{1}{12}\right) + (20)\left(\frac{1}{12}\right) \approx 14.83.$$

Hence, the contestant's expected accumulated winnings after the spin is smaller than the contestant's accumulated winnings before the spin.

d. If the contestant does not spin the wheel, then the contestant's wealth is certain to be $15,000. If the contestant spins the wheel, then the contestant's wealth is expected to be $14,833. Hence, if decisions are made on the basis of arithmetic expectation, the contestant should not spin the wheel. From this analysis, it is clear that a risk averse

3. Let Y_1 and Y_2 be the respective payoffs.

a. The sample space for this experiment is
$S = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}$.

b. The probability mass functions are given by:

$$p_{Y_1}[-3] = \frac{1}{8},\ p_{Y_1}[-1] = \frac{3}{8},\ p_{Y_1}[1] = \frac{3}{8},\ p_{Y_1}[3] = \frac{1}{8}$$

and

$$p_{Y_2}[-6] = \frac{1}{8},\ p_{Y_2}[1] = \frac{3}{8},\ p_{Y_2}[2] = \frac{3}{8},\ p_{Y_2}[3] = \frac{1}{8}.$$

c. The expected values for Y_1 and Y_2 are $E[Y_1] = 0$ and $E[Y_2] = \frac{3}{4}$, and the probabilities of a positive payoff are $\Pr[Y_1 > 0] = \frac{1}{2}$ and $\Pr[Y_2 > 0] = \frac{7}{8}$. These observations suggest that payoff 2 is preferable. Note, however, that the largest loss on payoff 2, when it occurs, is twice as large as the largest loss on payoff 1, when it occurs. Consequently, if you could not afford to lose more than \$3, you probably wouldn't choose payoff 2, despite its relatively superior payoff profile.

5. From the given formula for p_X, the random variable X has probability masses of $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ at the points $x = -1,\ 0,\ 1$ respectively, i.e.,

$$p_X[-1] = \frac{1}{4},\ p_X[0] = \frac{1}{2},\ p_X[1] = \frac{1}{4}.$$

a. The distribution function of X can be determined using the definition $F_X[x] = \Pr[X \le x]$ and considering separately the cases $x < -1$, $x = -1$, $-1 < x < 0$, $x = 0$, $0 < x < 1$, $x = 1$, $x > 1$. When we do this, we find that

$$F_X[x] = 0 \text{ for } x < -1,\ F_X[x] = \frac{1}{4} \text{ for } -1 \le x < 0,$$

$$F_X[x] = \frac{3}{4} \text{ for } 0 \le x < 1,\ \ F_X[x] = 1 \text{ for } x \ge 1.$$

b. From the definition of expected value, we have

$$E[X] = \sum x\, p_X[x] \ = \ (-1)\left(\frac{1}{4}\right) + (0)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{4}\right) \ = \ 0.$$

We could also have reached this conclusion by noting that the probability mass function is symmetric about $x = 0$.

6. a. The given distribution function is a step function with jumps at $x = -1$ and $x = \frac{2}{3}$. Hence, the only values of x for which $p_X[x] > 0$ are $x = -1$ and $x = \frac{2}{3}$. To see why this is so, consider, for example, the point $x = 0$. From the definition of F_X, $F_X[0] = \frac{1}{3}$ and $F_X[-1] = \frac{1}{3}$. However,

$$F_X[0] = \Pr[X \le 0] = \Pr[X \le -1] + \Pr[-1 < X < 0] + \Pr[X = 0] = F_X[-1] + \Pr[-1 < X < 0] + p_X[0].$$

Hence, $\Pr[-1 < X < 1] + p_X[0] = 0$. So, since probabilities cannot be negative, we have $p_X[0] = 0$ as claimed. One can show in a similar way that $p_X[x] = 0$ for $x \neq -1, \frac{2}{3}$.

Now

$$p_X\left[\frac{2}{3}\right] = \Pr\left[X = \frac{2}{3}\right] = \Pr\left[X \le \frac{2}{3}\right] - \Pr\left[X < \frac{2}{3}\right] = F_X\left[\frac{2}{3}\right] - \Pr\left[X < \frac{2}{3}\right].$$

Further, since $p_X[x] = 0$ for $-1 < x < \frac{2}{3}$, as demonstrated earlier, we have

$$\Pr\left[X < \frac{2}{3}\right] = \Pr[X \le -1] + \Pr\left[1 < X < \frac{2}{3}\right] = \Pr[X \le -1] = F_X[-1].$$

Hence,

$$p_X\left[\frac{2}{3}\right] = F_X\left[\frac{2}{3}\right] - F_X[-1] = 1 - \frac{1}{3} = \frac{2}{3}.$$

In particular, the value of $p_X\left[\frac{2}{3}\right]$ is the size of the jump on the graph of F_X at the point $x = \frac{2}{3}$. In a similar manner, we have $p_X[-1] = \frac{1}{3}$.

Consequently, the probability mass function of X is given by

$$p_X[-1] = \frac{1}{3},\ p_X\left[\frac{2}{3}\right] = \frac{2}{3},\ \text{and } p_X[x] = 0 \ \text{ for all other } x.$$

b. From the probability mass function determined in part a, the expected value is given by

$$E[X] = (-1)\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{1}{9}.$$

c. Using the observations given in part a, we have

$$\Pr\left[X < \frac{2}{3}\right] = \Pr[X \le -1] + \Pr\left[1 < X < \frac{2}{3}\right] = F_X[-1] + 0 = \frac{1}{3}.$$

7. We are given that $p_X[x] = \frac{1}{2}\, p_X[x-1]$ for $x = 1, 2, \ldots$ and that $p_X[x] = 0$ for $x \neq 0, 1, 2, \ldots$. Hence, we need only determine values of $p_X[x]$ for $x = 0, 1, 2, \ldots$. From the recurrence relationship, we have

$$p_X[n] = \left(\frac{1}{2}\right)^n p_X[0], \quad \text{for } n = 1, 2, \ldots .$$

Since $\sum_{n=0}^{\infty} p_X[n] = 1$, it follows that

$$p_X[0] \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1.$$

However, from the formula for the sum of a geometric series (i.e., $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $-1 < r < 1$), we have $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2$. Hence $p_X[0] = \frac{1}{2}$ and consequently,

$$p_X[n] = \left(\frac{1}{2}\right)^{n+1} \quad \text{for } n = 0, 1, 2, \ldots .$$

From this formula and the formula for the sum of a finite geometric series (i.e., $\sum_{n=0}^{k} a\, r^n = a\left(1 - r^{k+1}\right)/(1-r)$), we have

$$\Pr[X < 10] = \sum_{x=0}^{9} \left(\frac{1}{2}\right)^{x+1} = 1 - \left(\frac{1}{2}\right)^{10}$$

and similarly,

$$\Pr[X < n] = \sum_{x=0}^{n-1} \left(\frac{1}{2}\right)^{x+1} = 1 - \left(\frac{1}{2}\right)^{n}, \quad \text{for } n = 1, 2, 3, \ldots .$$

10. Let X_1 and X_2 be as defined in the question.

a. Put $Y_1 = X_1 + X_2$. Then $Y_1 = 2$ if the outcome is HH, $Y_1 = 1$ if the outcome is HT or TH, and $Y_1 = 0$ if the outcome is TT. Hence, the probability mass function for Y_1 is defined as follows:

$$p_{Y_1}[2] = \frac{1}{4}, \; p_{Y_1}[1] = \frac{1}{2}, \; p_{Y_1}[0] = \frac{1}{4}.$$

Note that Y_1 represents the number of heads obtained.

b. Put $Y_2 = X_1\, X_2$. Then $Y_2 = 1$ if the outcome is HH and $Y_2 = 0$ otherwise. Hence, the probability mass function for Y_2 is given by

$$p_{Y_2}[1] = \frac{1}{4},\ p_{Y_2}[0] = \frac{3}{4}.$$

Note that Y_2 is an indicator for the event that both coins land heads.

c. Put $Y_3 = 2\,X_1$. Then $Y_3 = 2$ if the outcome is HH or HT, and $Y_3 = 0$ if the outcome is TH or TT. Hence, the probability mass function for Y_3 is given by

$$p_{Y_3}[2] = \frac{1}{2},\ p_{Y_3}[0] = \frac{1}{2}.$$

From part a, it is clear that the distributions of Y_1 and Y_3 are different. Hence $X_1 + X_2 \nsim 2\,X_1$. Note, however, that $X_1 \sim X_2$ because both coins have the same probability of landing heads. Hence $X_1 + X_2 \nsim 2\,X_1$ even though $X_1 \sim X_2$.

d. Put $Y_4 = X_1^2$. Then $Y_4 = 1$ if the outcome is HH or HT, and $Y_4 = 0$ if the outcome is TH or TT. Hence, the probability mass function for Y_4 is given by

$$p_{Y_4}[1] = \frac{1}{2},\ p_{Y_4}[0] = \frac{1}{2}.$$

From part b, it is clear that the distributions of Y_2 and Y_4 are different. Hence, $X_1\,X_2 \nsim X_1^2$ even though $X_1 \sim X_2$.

e. From parts c and d, it follows that substitution of equivalent random variables is not valid. In particular, if $X_1 \sim X_2$, we cannot conclude that $X_1 + X_2 \sim 2\,X_1$ and we cannot conclude that $X_1\,X_2 \sim X_1^2$. On the other hand, since random variables are functions and the substitution rules are valid for functions, it follows that substitution of *equal* random variables is valid. In particular, if $X_1 = X_2$, then we can conclude that $X_1 + X_2 = 2\,X_1$ and that $X_1\,X_2 = X_1^2$. This issue is discussed in greater detail in section 4.1.11 of the textbook.

13. The desired probabilities are:

$$S[5] = \text{Exp}\left[-1^3\right] \approx .3679,$$
$$S[10] = \text{Exp}\left[-2^3\right] \approx .0003.$$

15. a. From the general relationship

$$F_X[x] = \int_{-\infty}^{x} f_X[s]\, ds$$

(see section 2.3 in the textbook) and the given form of the density function, it follows that

$$F_X[x] = 1 - \frac{1}{x^2}\ \text{for}\ x \geq 1,\ F_X[x] = 0\ \text{for}\ x < 1.$$

Indeed, for $x \geq 1$,

$$F_X[x] = \int_0^x \frac{2}{s^3}\, ds = \left(-s^{-2}\right)\big|_1^x = 1 - \frac{1}{x^2}.$$

b. From the given formula for f_X and the formula for the expectation of a continuous random variable (section 2.3 in the textbook), it follows that

$$E[X] = \int_{-\infty}^{\infty} x\, f_X[x]\, dx = \int_1^{\infty} x \frac{2}{x^3}\, dx = -2\, x^{-1}\big|_1^{\infty} = 2.$$

Note that we obtain the same answer, as we should, using the formula $E[X] = \int_0^{\infty} S_X[x]\, dx$. Indeed, since $S_X[x] = 1/x^2$ for $x \geq 1$ and $S_X[x] = 1$ for $x < 1$, we have

$$E[X] = 1 + \int_1^{\infty} \frac{1}{x^2}\, dx = 2.$$

c. From the formula for F_X determined in part b, we have

$$\Pr[X > 4] = 1 - \Pr[X \leq 4] = 1 - F_X[4] = 1 - \left(1 - \frac{1}{16}\right) = \frac{1}{16}.$$

16. a. From the general relationship $f_X[x] = F_X'[x]$ for continuous random variables (see section 2.3 in the textbook) we have

$$f_X[x] = \frac{3}{x^4} \text{ for } x \geq 1, \quad f_X[x] = 0 \text{ otherwise}.$$

(Strictly speaking, f_X is not defined at $x = 1$. However, if we assume that $f_X[1]$ represents the derivative of F_X from the right at $x = 1$, then $f_X[1] = 3$.)

b. From the formula for the expectation of a continuous random variable and the answer to part a, we have

$$E[X] = \int_{-\infty}^{\infty} x\, f_X[x]\, dx = \int_1^{\infty} x \frac{3}{x^4}\, dx = \frac{3}{2}.$$

c. From the given formula for F_X and the fact that $\Pr[X = a] = 0$ for all points a when X has a continuous distribution, we have

$\Pr[2 < X < 4] =$

$$\Pr[X \le 4] - \Pr[X \le 2] - \Pr[X = 4] = F_X[4] - F_X[2] - 0 = \left(1 - \frac{1}{64}\right) - \left(1 - \frac{1}{8}\right) = \frac{7}{64}.$$

Alternatively, the desired probability can be calculated by integrating the density function determined in part a over the region $2 < x < 4$:

$$\Pr[2 < X < 4] = \int_2^4 \frac{3}{x^4}\, dx = \frac{7}{64}.$$

23. Let V be the value of the account after 2 days.
    a. Yes. See answer to c,d below.

b. There are four possible outcomes: gains on both days, losses on both days, a gain followed by a loss, or a loss followed by a gain. Since gains and losses on different days are independent, it follows that

$V = (1.50)^2$ with probability $\frac{1}{4}$,

$V = (1.50)\,(0.60)$ with probability $\frac{1}{2}$,

$V = (0.60)^2$ with probability $\frac{1}{4}$.

Hence, the probability mass function for V is given by

$$p_V[0.36] = \frac{1}{4},\ p_V[0.90] = \frac{1}{2},\ p_V[2.25] = \frac{1}{4},\ p_V[v] = 0 \text{ otherwise}.$$

c. From the answer to part b, we have

$$\Pr[V > 1] = \frac{1}{4}$$

and

$$E[V] = (2.25)\left(\frac{1}{4}\right) + (0.90)\left(\frac{1}{2}\right) + (0.36)\left(\frac{1}{4}\right) = 1.1025.$$

Hence, there is only a 25% chance of our coming out ahead after two days. This agrees with our observation in part a. The fact that $E[V] > 1$ may lead one to believe that the investment is a good one. However, this is only true in certain circumstances, as explained in the next part of the question.

d. From part c, $E[V] = 1.1025 > 1$. Since $E[V]$ represents the average accumulation *per investment* for a large number of independent investments of the type described, it follows that the investment opportunity is a good one if we can make a large number of independent investments of this type. This is so even though only one quarter of the investments will be profitable ($\Pr[V > 1] = .25$ from part c) because the gains, when they occur, more than make up for the losses on the other three quarters of the investments. Note the importance of the assumption that investment returns on the indvidual