Department of Mathematics, University of Houston
Math 3338 - Probability - David Blecher
Key to First Midterm Mock Exam Part 2.

The other mock exam had no questions on continuous distributions, densities, or many questions on Section 4.1. In addition, Questions 3 and 5 there were perhaps a bit too easy, although they are good practice problems (you can expect a Venn diagram problem similar to some done in the class and Homework 2, but harder than Question 5 on the other mock exam). In addition, there could be questions on 4.1 like those in the homework, or like the following:

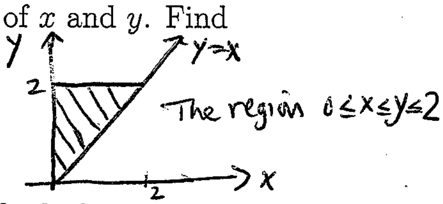
1. The random variable X of the life of a brand of battery has density $f_X(x) = \frac{1}{2}e^{-\frac{x}{2}}$, for $x > 0$. We are working in units of hundreds of hours. What is the probability that the battery life is less than 200 or greater than 400 hours? If we knew that the battery had already been used for 200 hours, what is the probability it lasts over 400 hours?

   Solution: The answer to the first question is $\int_0^2 \frac{1}{2}e^{-\frac{x}{2}}\,dx + \int_4^\infty \frac{1}{2}e^{-\frac{x}{2}}\,dx = 1 - e^{-1} + e^{-2}$. The answer to the second question is $Pr(X > 4|X > 2) = Pr(X > 4)/Pr(X > 2) = e^{-2}/(1 - e^{-1}) = e^{-1}/(e-1)$.

2. Random variables X and Y are independent and identically distributed, with common density $f(x) = e^{-x}$ for $x > 0$. Find $\Pr(Y > 2X)$.

   Solution: By a formula in the classmotes and text, this is the integral over the region $\{(x, y) : y > 2x\}$ of the joint pdf $f_{XY}(x, y) = e^{-x}e^{-y}$. We get $\int_0^\infty \int_0^{y/2} e^{-x}e^{-y}dxdy$. The inside integral is $-e^{-x}e^{-y}\,|_0^{x=y/2} = -e^{-y/2}e^{-y} + e^{-y}$, so the final answer is $\int_0^\infty (-e^{-3y/2} + e^{-y})\,dy = (\frac{2}{3}e^{-3y/2} - e^{-y})|_0^\infty = 1 - \frac{2}{3} = \frac{1}{3}$.

3. A soft drink machine has a random supply Y gallons at the beginning of a given day, and dispenses a random amount X gallons during the day. It has been observed that the joint density of X and Y is $1/2$ if $0 \leq x \leq y \leq 2$, and is 0 for other values of x and y. Find
   (a) The marginal densities of X and Y,
   (b) The probability that $Y < 1/2$ on a given day,
   (c) The conditional density $f_{X|Y=1}$,
   (d) $\Pr(X < \frac{1}{2}\,|\,Y = 1)$.

   You may use any formulae from the classnotes (in the real test, difficult formulae will be given).

   Solution: Note $f_{X,Y}(x, y) = 1/2$ if (x, y) is in the shaded triangle shown, and $f_{X,Y}(x, y) = 0$ if (x, y) is not in the triangle. One needs to look at this triangle to get some of the points below:

   (a) $f_Y(y) = \int_{-\infty}^\infty f_{X,Y}(x, y)\,dx = \int_0^y \frac{1}{2}\,dx = y/2$ if $0 \leq y \leq 2$. For other values of y we have $f_{X,Y} = 0$ and so $f_Y(y) = 0$.

   Similarly, $f_X(x) = \int_{-\infty}^\infty f_{X,Y}(x, y)\,dy = \int_x^2 \frac{1}{2}\,dy = \frac{2-x}{2}$, if $0 \leq x \leq 2$. For other values of x we have $f_{X,Y} = 0$ and so $f_X(x) = 0$.

   (b) $\Pr(Y < 1/2) = F_Y(1/2) = \int_0^{1/2} f_Y(y)dy = \int_0^{1/2} \frac{y}{2}\,dy = 1/16$.

   (c) $f_{X|Y=1}(x) = f_{X,Y}(x, 1)/f_Y(1) = \frac{1/2}{1/2} = 1$ if $0 \leq x \leq 1$, and is 0 otherwise.

   (d) $\Pr(X < \frac{1}{2}\,|\,Y = 1) = \int_0^{\frac{1}{2}} f_{X|Y=1}(x)dx = \int_0^{\frac{1}{2}} 1dx = \frac{1}{2}$.