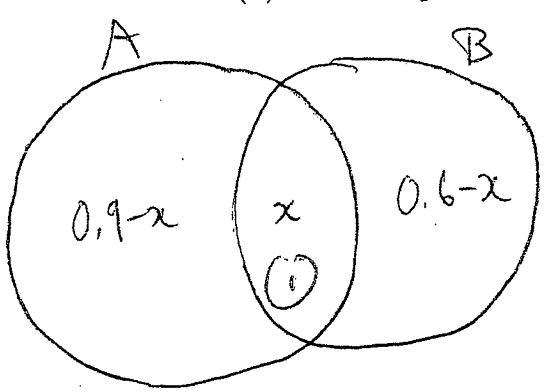
KEY, TEST 1 MATH 3338, F2009

1. (10pts) Every house in my neighborhood has either a cellphone or a computer or both. Let A be the event that a house has a cellphone, and B the event that it has a computer. According to a neighborhood survey, $P(A) = 0.9, P(B) = 0.6$.

   (a) What is the probability that a house has a computer but not a cellphone?

   (b) Find the probability $P(A|B)$.



1 1

$$1 = (0.9 - x) + x + (0.6 - x) = 1.5 - x \Rightarrow x = 0.5$$

(a) $\Pr(B \setminus A) = 0.6 - 0.5 = \boxed{0.1}$ 3

(b) $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} = \dfrac{0.5}{0.6} = \boxed{5/6}$ 4

5/6

2. (10pts) A biased coin is tossed 5 times. The probability of getting a head on any one toss is 0.4. Determine the probability that exactly two heads are obtained.

Soln. $\boxed{\binom{5}{2}(0.4)^2(0.6)^3}$.

Hint: On this page you may want to use the law of total probability and/or Bayes theorem.

3. (15pts) We have two boxes. The first contains 3 red and 2 blue marbles, the second box contains 2 red and 8 blue marbles. A fair coin is tossed, if it is 'heads' we take a marble from the first box, if it is 'tails' we take a marble from the second box. What is the probability that this procedure results in a red marble? [Hint: first ask questions like: What is the probability of drawing a red marble from the first box, etc.]

4

R = getting a red marble, $B = R^c$ = getting a blue marble

F = first box, S = second box

$$Pr(R) = Pr(R|F)Pr(F) + Pr(R|S)Pr(S)$$

$$= \frac{3}{5} \cdot \frac{1}{2} + \frac{2}{10} \cdot \frac{1}{2} = \frac{4}{5 \cdot 2} = \boxed{\frac{2}{5}}$$

2 2 1 2 2 2

4. (15pts) Suppose that the procedure in the last question results in a blue marble. What is the probability that it came from the second box?

We want $Pr(S|B) = \frac{Pr(B|S)\,Pr(S)}{Pr(B)}$ (Bayes) 3

2

$$= \frac{\frac{8}{10} \cdot \frac{1}{2}}{1 - Pr(R)} = \frac{2/5}{1 - 2/5} = \boxed{\frac{2}{3}}$$

2 2 2 1 3

5. (30pts) Roll a pair of fair four-sided dice; one red and one black. The outcome of each of the dice of course will be an integer between 1 and 4. Let X be the outcome on the red die, and Y the absolute value of the difference of the two outcomes.

(a) Construct a contingency table for the joint probability mass function of (X, Y). On your table show also the marginal probability mass functions of X and Y.

(b) Are X and Y independent? Are they identically distributed? Explain.

(c) Calculate the expected value of Y

(a)

Y \ X	1	2	3	4	p_Y
0	1/16	1/16	1/16	1/16	1/4
1	1/16	$\frac{1}{8}$	$\frac{1}{8}$	1/16	3/8
2	1/16	1/16	1/16	1/16	1/4
3	1/16	0	0	$\frac{1}{16}$	1/8
p_X	1/4	$\frac{1}{4}$	1/4	1/4	

(For example, to get the circled $\frac{1}{8}$ note here $X=2$, so a 2 on red die, and $Y=1$, so a 1 or a 3 on the blue die. Now $Pr(X=2) = 1/4$, and $Pr(1 \text{ or } 3 \text{ on blue die}) = 2/4 = 1/2$, so $p_{X,Y}(2,1) = \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}$.)

(b) Not independent (2) since $p_X(x)\,p_Y(y) \neq p_{X,Y}(x,y)$ (2), for example

$$p_{X,Y}(2,3) = 0 \neq \frac{1}{4}\cdot\frac{1}{8} = p_X(2)\,p_Y(3).$$

Not identically distributed (2), since $p_X \neq p_Y$ (1)

(c) $E(Y) = \sum_y y\,p_Y(y) = 0\cdot\frac{1}{4} + 1\cdot\frac{3}{8} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{8} = \boxed{5/4}$

(1) (1) (1) (1) (2)

Grading for (a): give ($\frac{1}{2}$) points for each correct entry in the square ($16 \times \frac{1}{2} = 8$ points)

Give (1) point for each correct value for p_X and p_Y $(1\times 8 = 8)$

Give (1) point for Y values between 0 and 3. Total = 17 points

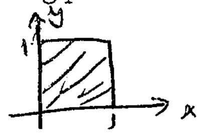
Give partial credit for students who did something different.

6. (30pts) One insurance clerk works on auto policies, another writes up homeowner policies. On a random day, let X be the fraction of the working day the first clerk writes up on auto policies, and Y the fraction of the day spent on homeowner policies. Suppose that the joint density of (X, Y) is $\frac{6}{5}(x + y^2)$, for $0 \le x \le 1$ and $0 \le y \le 1$.

   (a) Find the probability that they both spend less than half of the day writing policies.

   (b) Find the marginal density of X.

   (c) Find $Pr(X < 1/2)$.



Solution: (a)
$$Pr\left(X<\tfrac{1}{2}, Y<\tfrac{1}{2}\right) = \int_0^{1/2}\int_0^{1/2} \frac{6}{5}(x+y^2)\,dx\,dy$$
$$= \int_0^{1/2} \frac{6}{5}\left(\frac{x^2}{2} + y^2 x\right)\Big|_{x=0}^{x=1/2} dy$$
$$= \int_0^{1/2}\left(\frac{3}{20} + \frac{3}{5}y^2\right) dy = \left(\frac{3}{20}y + \frac{1}{5}y^3\right)\Big|_0^{1/2}$$
$$= \frac{3}{40} + \frac{1}{40} = \boxed{\frac{1}{10}}$$

(b)
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^1 \frac{6}{5}(x+y^2)\,dy = \boxed{\frac{6}{5}x + \frac{2}{5}}$$

(c)
$$Pr\left(X<\tfrac{1}{2}\right) = \int_{-\infty}^{1/2} f_X(x)\,dx = \int_0^{1/2}\left(\frac{6}{5}x + \frac{2}{5}\right)dx$$
$$= \left(\frac{3x^2}{5} + \frac{2}{5}x\right)\Big|_0^{1/2}$$
$$= \frac{3}{20} + \frac{1}{5} = \boxed{\frac{7}{20}}$$