Key - Test 2 - 3338 (This was 'open book') with 15 bonus points [Tot = 115%]

1. (20pts) A distribution has moment generating function $M(t) = \frac{1}{\sqrt{1-t}}$ for $t < 1$.

   (a) Find the cumulant generating function ψ. Simplify your answer.

   (b) Find the mean and variance and skewness for this distribution. Simplify.

Soln: (a) $\psi = \overset{\textcircled{1}}{\ln} M = \overset{\textcircled{1}}{\ln 1} - \overset{\textcircled{1}}{\ln (1-t)^{1/2}} = \overset{\textcircled{1}}{0} - \frac{1}{2}\ln(1-t) = \overset{\textcircled{1}}{\boxed{-\frac{1}{2}\ln(1-t)}}$

(b) $\psi' = \underset{\textcircled{1}}{\frac{1}{2}}\underset{\textcircled{1}}{(1-t)^{-1}}$, $\psi'' = +\underset{\textcircled{1}}{\frac{1}{2}}\underset{\textcircled{1}}{(1-t)^{-2}}$, $\psi''' = \underset{\textcircled{1}}{\cancel{\tfrac{1}{2}}\cancel{(2)}}\underset{\textcircled{1}}{(1-t)^{-3}} = (1-t)^{-3}$

$\underset{\textcircled{1}}{\psi'(0) = \frac{1}{2}}$, $\underset{\textcircled{1}}{\psi''(0) = \frac{1}{2}}$, $\underset{\textcircled{1}}{\psi'''(0) = 1}$

So $\underset{\textcircled{1}}{\text{mean}} = \underset{\textcircled{1}}{\text{variance}} = \underset{\textcircled{1}}{\frac{1}{2}}$, skewness $= \frac{1^{\textcircled{1}}}{\underset{\textcircled{1}}{\left(\frac{1}{2}\right)^{3/2}}} = \underset{\textcircled{1}}{2^{3/2}} = \sqrt{8}$

2. (15pts) From 6pm telephone solicitors call Ellen at an average rate of one call in every 4 minutes, according to a Poisson process. What is the probability that Ellen gets 3 or more calls by 6:32 pm? You do not need to compute the exponential appearing in your answer.

$\underline{X} = \#(\text{calls between 6 and 6:32}) \sim \overset{\textcircled{1}}{\text{Poisson}}\,(\overset{\textcircled{3}}{8})$

(1 call per 4 minutes) = (8 calls per 32 minutes)

$\underset{\textcircled{1}}{\Pr(X \geq 3)} = 1 - \Pr(X=0) - \Pr(X=1) - \Pr(X=2) \quad \textcircled{2}$

$\doteq 1 - \underset{\textcircled{1}}{e^{-8}} - \underset{\textcircled{2}}{8e^{-8}} - \underset{\textcircled{2}}{\frac{8^2}{2}e^{-8}}$

$= 1 - 41e^{-8} \quad \textcircled{3}$

3. (15pts) Dave keeps trying a telephone hotline until he is connected. Data has shown that 80 % of calls do not get connected, due to the popularity of the hotline.

   (a) What is the probability that Dave gets more than seven busy signals, before getting connected? You do not need to compute the power in your answer.

   (b) How many times should he expect to get a busy signal (before getting connected)?

Solution: (a) $X = \#(\text{busy signals}) \sim \overset{\textcircled{2}}{\text{Geom}}\,\overset{\textcircled{2}}{(0.2)}$ $(p = 1-0.8 = 0.2)$

$$\underset{\textcircled{1}}{\Pr(X > 7)} = \underset{\textcircled{1}}{\sum_{k=8}^{\infty} p_X(k)} = \underset{\textcircled{1}}{\sum_{k=8}^{\infty} p(1-p)^k} = \underset{\textcircled{2}}{\frac{p(1-p)^8}{1-(1-p)}} = \boxed{(0.8)^8}$$

(b) $E(X) = \frac{1-p}{p}^{\textcircled{2}} = \frac{0.8}{0.2}^{\textcircled{2}} = \boxed{4}$ ②

[Give partial credit on (a) if they say: $1-\sum_{k=0}^{7} p(1-p)^k = 1-\cancel{p}\frac{1-(1-p)^8}{\cancel{1-(1-p)}}$
$= (1-p)^8$]
Or something like this.

4. (20pts) X and Y are independent random variables, each with distribution Geom($\frac{3}{4}$). Let $S = X + Y$.

   (a) Identify the distribution of S, and its parameters.

   (b) Compute $\Pr(S = 2)$.

   (c) Does S have the same distribution as $2X$? Explain, by looking at $p_{2X}(2)$.

Soln: (a) $S \sim \underset{\textcircled{3}}{\text{NegBin}}\,(\underset{\textcircled{1}}{2}, \underset{\textcircled{1}}{\tfrac{3}{4}})$

(b) $p_S(2) = \underset{\textcircled{2}}{\binom{3}{1}}\underset{\textcircled{2}}{\left(\frac{3}{4}\right)^2}\underset{\textcircled{2}}{\left(\frac{1}{4}\right)^2} = \underset{\textcircled{2}}{3 \cdot \frac{9}{16} \cdot \frac{1}{16}} = \frac{27}{256}$

(c) $p_{2X}(2) = \Pr(\underset{\textcircled{1}}{2X = 2}) = \Pr(\underset{\textcircled{1}}{X = 1}) = \underset{\textcircled{2}}{\frac{3}{4}} \cdot \underset{\textcircled{1}}{\frac{1}{4}} = \frac{3}{16}$

Since this differs from $p_S(2)$, S and $2X$ have different distributions ②

5. (15pts) Let X be a random variable whose moments $E(X^k) = \frac{4}{5}$ for $k = 1, 2, \cdots$.

(a) Show that $M_X(t) = \frac{1}{5} + \frac{4}{5}e^t$.

(b) Find the pmf of X.

(a) $$M_X(t) \textcircled{1} = \overset{\textcircled{1}}{1} + \overset{\textcircled{1}}{\sum_{k=1}^{\infty}} \overset{\textcircled{1}}{E(X^k)} \overset{\textcircled{1}}{\frac{t^k}{k!}} = 1 + \overset{\textcircled{1}}{\sum_{k=1}^{\infty} \frac{4}{5} \frac{t^k}{k!}} = \overset{\textcircled{3}}{\frac{1}{5} + \frac{4}{5}\sum_{k=0}^{\infty} \frac{t^k}{k!}}$$

$$= \frac{1}{5} + \frac{4}{5} e^t \textcircled{1}.$$

An alternative solution to (a):

If $M = \frac{1}{5} + \frac{4}{5}e^t \rightarrow M_X' = M_X'' = M_X''' = \cdots = \frac{4}{5}e^t$

so $E(X^k) = M_X^{(k)}(0) = \frac{4}{5}$

since the moments determine the distribution, $M_X = M$.

(b) $$p_X(x) = \begin{cases} 1/5 \textcircled{1}, & x = 0 \textcircled{1} \\ 4/5 \textcircled{1}, & x = 1 \textcircled{1} \\ 0, & \text{else} \textcircled{1} \end{cases}$$

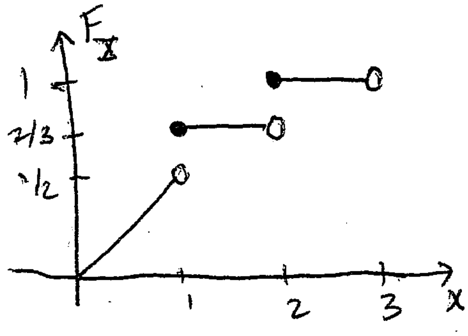
6. (30) (25pts) Consider the distribution function $F_X(x) = \begin{cases} x/2, \ 0 \le x < 1, \\ 2/3, \ 1 \le x < 2, \\ 1, \ x \ge 2. \end{cases}$ Find the mean and variance of the mixed random variable X.

Soln: $$E(X) = \underbrace{\overset{\textcircled{1}}{\int_0^1} \overset{\textcircled{1}}{x} \cdot \overset{\textcircled{1}}{\frac{1}{2}\,dx}}_{\textcircled{1}\ \text{continuous part}} + \underbrace{\overset{\textcircled{1}}{1} \cdot \overset{\textcircled{3}}{\frac{1}{6}} + \overset{\textcircled{1}}{2} \cdot \overset{\textcircled{2}}{\frac{1}{3}}}_{\text{discrete part (see graph below)}} = \overset{\textcircled{2}}{\frac{1}{4} + \frac{1}{6} + \frac{2}{3}} = \overset{\textcircled{2}}{\boxed{\frac{13}{12}}}$$

Similarly,

$$E(X^2) \textcircled{1} = \overset{\textcircled{1}}{\int_0^1} x^2 \cdot \overset{\textcircled{1}}{\frac{1}{2}\,dx} + \overset{\textcircled{1}}{1^2} \cdot \overset{\textcircled{2}}{\frac{1}{6}} + \overset{\textcircled{1}}{2^2} \cdot \frac{1}{3} = \overset{\textcircled{2}}{\frac{1}{6} + \frac{1}{6} + \frac{4}{3}} = \overset{\textcircled{2}}{\boxed{\frac{5}{3}}}$$

$$\text{So } Var(X) = \underset{\textcircled{1}}{E(X^2)} - \underset{\textcircled{1}}{E(X)^2} = \boxed{\frac{5}{3} - \left(\frac{13}{12}\right)^2} \textcircled{2}$$



← Note F_X has a jump of $\frac{1}{6}$ at $x=1$ and a jump of $\frac{1}{3}$ at $x=2$