NAME: ______________________________

Department of Mathematics, University of Houston
Math 3333 - Intermediate Analysis - David Blecher
Mock Final exam.

Instructions. Time= 3 hours. THIS IS MUCH LONGER THAN THE REAL FINAL. Put all books and papers at the side of the room. Show all working and reasoning, the points are almost all for logical, complete reasoning. [Approximate point values are given.]

1. Write the negation of the statement: $\forall \epsilon > 0$, $\exists\, \delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$ and $x \in D$. [5]

2. Prove that between every two real numbers, there is a rational number. [13]

3. (a) What is a boundary point of a set S as defined in the notes? [3]
   (b) What does it mean for a set to be open (as defined in the notes)? [2]
   (c) Prove that a set S is open if and only if for any $x \in S$ there exists a $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset S$. [7]

4. (a) Prove that if S is bounded above then $\sup(S)$ is a boundary point of S. [10]
   (b) Prove that a closed set S which is bounded above has a maximum. [5]

5. Let $S = \{1 + \frac{1}{n} : n \in \mathbb{N}\}$.
   (a) Prove that $1 = \inf S$ (give all details). [7]
   (b) Find all boundary points of S, and prove (using an ϵ argument) that the smallest of these numbers is a boundary point. [7]
   (c) Is this set open? Why? Is it closed? Why? [4]

6. (a) Define what we mean by $\lim_{n\to\infty} s_n = s$ (the definition involving ϵ). [4]
   (b) Prove that if $s_n \to s$ and $t_n \to t \neq 0$, then $\frac{s_n}{t_n} \to \frac{s}{t}$. [12]
   (c) Prove that a decreasing bounded sequence converges to its infimum. [8]

7. Prove that $\lim_n \frac{n^2-2}{n^2+2n+2} = 1$. [8]

8. (a) If (s_n) is a sequence, then what is a *subsequence* of (s_n)? [4]
   (b) State the Bolzano-Weierstrass theorem for sequences. [4]
   (c) Complete the sentence: "A number x is in the closure $\bar{S}$ of a set S iff for every $\epsilon > 0$, the intersection of S with ....". [3]

9. Using the ϵ-δ definition, show that $\lim_{x\to 1} \frac{x^2-4}{x-4} = 1$. [15]

Solution: %

10. Suppose that $f : (a, b) \to \mathbb{R}$, $g : (a, b) \to \mathbb{R}$, $L \in \mathbb{R}$, and $a < c < b$.
    (a) State and prove our 'Main Theorem #1' (a criterion for when $\lim_{x\to c} f(x) = L$ in terms of sequences converging to c). [27]
    (b) If $C \le f(x) \le D$ for all x, and if $\lim_{x\to c} f(x) = L$, prove $C \le L \le D$. [6]

11. Define what it means for f to be continuous at c, and write down three other equivalent conditions. [11]

12. (a) State the intermediate value theorem. [6]
    (b) Let $K \ge 1$ and consider the function $f(x) = x^2 - K$. By looking at $f(0)$ and $f(K)$, and using the IVT, show that $\sqrt{K}$ exists. [5]

13. If $f, g : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, prove that the product $f(x)g(x)$ is differentiable at c. [8]

14. (a) State the mean value theorem. [6]
    (b) Prove that if $f'(x) = g'(x)$ for all $x \in (a, b)$ then $f(x) = g(x) + C$ on (a, b) for a constant C. [4]

15. (a) State Riemann's condition for integrability. [4]
    (b) What is a uniformly continuous function? [4]
    (c) Prove that a continuous function $f : [a, b] \to \mathbb{R}$ is integrable. [15]

16. State and prove the second fundamental theorem of calculus (about integrating a derivative). [5+18]

17. (a) What does it mean for a series of real numbers to be convergent? Absolutely convergent?
(b) State the Cauchy criterion for convergence of a series of real numbers.
(c) Prove that an absolutely convergent series of real numbers converges and then $|\sum_{k=1}^{\infty} a_k| \leq \sum_{k=1}^{\infty} |a_k|$.
(d) Show that $\sum_{k=2}^{\infty} \frac{(-1)^k}{k(\log k)^2}$ converges, and converges absolutely.
(e) What is the 'tail' of a series? Show that if a series converges then its tail converges. [5 points]
(f) Complete the sentence: "A nonnegative series converges iff the sequence ______________ is ______________ above, and then the sum of the series equals the ______________". [3 points]