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Math 3333 - Intermediate Analysis - David Blecher
Final exam–August 2010.

Instructions. Time= 3 hours. Put all books and papers at the side of the room. Show all working and reasoning, the points are almost all for logical, complete reasoning. [Approximate point values are given, total = approximately 200 points plus 42 bonus points].

1. Write the negation of the statement: $\forall \epsilon > 0$, $\exists\, \delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$ and $x \in D$. [5]

   Solution: $\exists \epsilon > 0$ s.t. $\forall \delta > 0$, $\exists\, x \in D$ s.t. $0 < |x - c| < \delta$, but $|f(x) - L| \geq \epsilon$. [5]

2. Prove that between every two real numbers, there is a rational number. [13]

   Solution: Suppose that $x < y$. By the Archimidean principle, choose $n \in \mathbb{N}$ with $n(y-x) > 1$. So the distance between nx and ny is > 1. Thus by a lemma in class there must exist $m \in \mathbb{Z}$ with $nx < m < ny$. Dividing by n we have $x < \frac{m}{n} < y$.

3. (a) What is a boundary point of a set S as defined in the notes? [3]
   (b) What does it mean for a set to be open (as defined in the notes)? [2]
   (c) Prove that a set S is open if and only if for any $x \in S$ there exists a $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset S$. [7]

   Solution. (a) This is a number x such that for every $\epsilon > 0$, we have $(x - \epsilon, x + \epsilon) \cap S \neq \emptyset$ and $(x - \epsilon, x + \epsilon) \cap S^c \neq \emptyset$. [3]

   (b) That it contains none of its boundary points. [2]

   (c) Suppose $x \in S$. Then to say that there exists a $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset S$, is the same as saying that x is not a boundary point for S. (because a x is a boundary point for S iff for every $\epsilon > 0$, we have $(x - \epsilon, x + \epsilon) \cap S^c \neq \emptyset$, and the negation of this is that there exists a $\epsilon > 0$, such that $(x - \epsilon, x + \epsilon) \subset S$.) Saying S is open is the same as saying that S contains none of its boundary points, and this is the same as saying that for any $x \in S$, x is not a boundary point. Thus saying S contains none of its boundary points is the same as saying that for any $x \in S$, there exists a $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset S$. [7]

4. (a) Prove that if S is bounded above then $\sup(S)$ is a boundary point of S. [10]
   (b) Prove that a closed set S which is bounded above has a maximum. [5]

   Solution. (a) Let $\alpha = \sup S$, and let $\epsilon > 0$ be given. Since α is an upper bound of S, the interval $(\alpha, \alpha + \epsilon)$ contains no points of S, and hence contains points of S^c. On the other hand, $\alpha - \epsilon$ is not an upper bound of S, so the interval $(\alpha - \epsilon, \alpha)$ must contain a point in S. Thus the interval $(\alpha-\epsilon, \alpha+\epsilon)$ contains both points of S^c and points of S. So $\alpha \in bd(S)$.

   (b) By (a), $\sup(S) \in \text{Bdy}(S) \subset S$. So S has a maximum.

5. Let $S = \{1 + \frac{1}{n} : n \in \mathbb{N}\}$.
   (a) Prove that $1 = \inf S$ (give all details). [7]
   (b) Find all boundary points of S, and prove (using an ϵ argument) that the smallest of these numbers is a boundary point. [7]
   (c) Is this set open? Why? Is it closed? Why? [4]

Solution. (a) Certainly 1 is a lower bound. However, if $\alpha > 1$ then $\alpha - 1 > 0$, so by the Archimidean property we can find $n \in \mathbb{N}$ with $\frac{1}{n} < \alpha - 1$. This implies that $1 + \frac{1}{n} < \alpha$, so that α is not a lower bound of S. So $1 = \inf S$. [7]

(b) The boundary points are $S \cup \{1\}$. To see that 1 is a boundary point, if $\epsilon > 0$ is given notice that $(1 - \epsilon, 1 + \epsilon)$ contains points in S^c (e.g. 1), and points in S, by the argument in (a) (taking $\alpha = 1 + \epsilon$). [7]

(c) It is not open, since it contains some of its boundary points. It is not closed since it does not contain the boundary point 1. [4]

6. (a) Define what we mean by $\lim_{n\to\infty} s_n = s$ (the definition involving ϵ). [4]
   (b) Prove that if $s_n \to s$ and $t_n \to t \neq 0$, then $\frac{s_n}{t_n} \to \frac{s}{t}$. [12]
   (c) Prove that a decreasing bounded sequence converges to its infimum. [8]

Solution. (a) That given any $\epsilon > 0$, $\exists N$ such that $|s_n - s| < \epsilon$ whenever $n \geq N$. [4]

(b) Note that

$$\left|\frac{s_n}{t_n} - \frac{s}{t}\right| = \left|\frac{s_n t - t_n s}{t_n t}\right| = \frac{|s_n t - st + st - t_n s|}{|t_n||t|} \leq \frac{|s_n t - st| + |st - t_n s|}{|t_n||t|} = \frac{|s_n - s||t| + |s||t - t_n|}{|t_n||t|}.$$

By Fact 8, $\exists N$ s.t. $|t_n| > |t|/2$ for $n \geq N$. Thus for $n \geq N$,

$$\left|\frac{s_n}{t_n} - \frac{s}{t}\right| \leq \frac{|s_n - s||t| + |s||t - t_n|}{|t_n||t|} \leq \frac{2}{|t|^2}(|s_n - s||t| + |s||t - t_n|).$$

Since $|s_n - s| \to 0$ and $|t - t_n| \to 0$, as $n \to \infty$, by Fact 3 (or another part of Fact 9) it follows that $|s_n - s||t| + |s||t - t_n| \to 0$ too. By Fact 3 again, $\frac{2}{|t|^2}(|s_n - s||t| + |s||t - t_n|) \to 0$ as $n \to \infty$. Thus we conclude from Fact 6, that $\frac{s_n}{t_n} \to \frac{s}{t}$ as $n \to \infty$. [12]

(c) If (s_n) is an decreasing bounded sequence, then $\{s_n : n = 1, 2, \cdots\}$ has a greatest lower bound m say. Since $m + \epsilon$ is not a lower bound there exists N with $s_N < m + \epsilon$. If $n \geq N$ then $M + \epsilon > s_N \geq s_n \geq m > m - \epsilon$. Thus $s_n \to m$. [8]

7. Prove that $\lim_n \frac{n^2-2}{n^2+2n+2} = 1$. [8]

Solution. $\frac{n^2-2}{n^2+2n+2} - 1 = \frac{-4-2n}{n^2+2n+2} = -2\frac{n+2}{n^2+2n+2}$. Thus

$$\left|\frac{n^2 - 2}{n^2 + 2n + 2} - 1\right| = 2\frac{n + 2}{n^2 + 2n + 2} \leq 2\frac{n + 2}{n^2} = 2(\frac{1}{n} + \frac{2}{n^2} \to 0.$$

Therefore $\lim_n \frac{n^2-2}{n^2+2n+2} = 1$. [8]

8. (a) If (s_n) is a sequence, then what is a *subsequence* of (s_n)? [4]
(b) State the Bolzano-Weierstrass theorem for sequences. [4]
(c) Complete the sentence: "A number x is in the closure $\bar{S}$ of a set S iff for every $\epsilon > 0$, the intersection of S with ....". [3]

Solution: (a) It is a sequence (s_{n_k}), where $n_k \in \mathbb{N}$ with $n_1 < n_2 < n_3 < \cdots$.

(b) Every bounded sequence has a convergent subsequence. [3]

(c) ... with $(x-\epsilon, x+\epsilon)$ is not empty.

9. Using the ϵ-δ definition, show that $\lim_{x\to 1} \frac{x^2-4}{x-4} = 1$. [15]

Solution: We have

$$\left|\frac{x^2-4}{x-4} - 1\right| = \frac{|x^2-4-(x-4)|}{|x-4|} = \frac{|x^2-x|}{|x-4|} = \frac{|x||x-1|}{|x-4|}.$$

If $|x-1| < 1$ then $0 < x < 2$ and $-4 < x-4 < -2$ so that $|x-4| > 2$. Thus $\frac{|x||x-1|}{|x-4|} < \frac{2|x-1|}{2} = |x-1|$. So given $\epsilon > 0$ choose $\delta < \epsilon$ and $\delta < 1$. If $0 < |x-c| < \delta$ then by the calculations above,

$$\left|\frac{x^2-4}{x-4} - 1\right| = \frac{|x||x-1|}{|x-4|} < |x-1| < \delta < \epsilon.$$

10. Suppose that $f : (a,b) \to \mathbb{R}$, $g : (a,b) \to \mathbb{R}$, $L \in \mathbb{R}$, and $a < c < b$.
(a) State and prove our 'Main Theorem #1' (a criterion for when $\lim_{x\to c} f(x) = L$ in terms of sequences converging to c). [27]
(b) If $C \leq f(x) \leq D$ for all x, and if $\lim_{x\to c} f(x) = L$, prove $C \leq L \leq D$. [6]

Solution: (a) $\lim_{x\to c} f(x) = L$ iff whenever (s_n) is a sequence in $(a,b)\setminus\{c\}$ with $s_n \to c$ then $f(s_n) \to L$.

Suppose that $\lim_{x\to c} f(x) \neq L$. Thus $\exists \epsilon > 0$ such that $\forall \delta > 0$, $\exists x \in (c-\delta, c+\delta)$ such that $x \neq c$ and $|f(x) - L| \geq \epsilon$. Taking $\delta = \frac{1}{n}$, there exists $s_n \in (c - \frac{1}{n}, c + \frac{1}{n})$ such that $s_n \neq c$ and $|f(s_n) - L| \geq \epsilon$. Clearly $s_n \to c$ by 'squeezing/pinching' fact about sequences, but $f(s_n)$ does not converge to L. [10]

Conversely, suppose that $\lim_{x\to c} f(x) = L$. That is, given $\epsilon > 0$ $\exists\, \delta > 0$ s.t. $|f(x) - L| < \epsilon$ whenever $0 < |x-c| < \delta$. If $s_n \to c$, $s_n \neq c$ then $\exists N$ s.t. $0 < |s_n - c| < \delta$ whenever $n \geq N$. So if $n \geq N$ then $|f(s_n) - L| < \epsilon$, and so $f(s_n) \to f(c)$. [10]

(b) Let (s_n) be a sequence in (c,b) with $s_n \to c$. Then $f(s_n) \to L$ by (a). However $C \leq f(s_n) \leq D$, so by a Fact from the Sequences handout, $C \leq L \leq D$.

11. Define what it means for f to be continuous at c, and write down three other equivalent conditions. [11]

Soln. For every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x-c| < \delta$.

An equivalent condition: $\lim_{x\to c} f(x) = f(c)$. Another is: $f(s_n) \to f(c)$ whenever $s_n \to c$, $s_n \in (a,b)$. Another is: Given a nhd U of $f(c)$, there exists a nhd V of c with $f(V \cap (a,b)) \subset U$.

12. (a) State the intermediate value theorem. [6]
(b) Let $K \geq 1$ and consider the function $f(x) = x^2 - K$. By looking at $f(0)$ and $f(K)$, and using the IVT, show that $\sqrt{K}$ exists. [5]

Soln. (a) If $f : [a, b] \to \mathbb{R}$ is continuous, and if z is a number between $f(a)$ and $f(b)$ then there exists a $c \in (a, b)$ with $f(c) = z$. [6]

(b) $f(0) = -K < 0$. $f(K) = K^2 - K > 0$. So by the IVT there exists a $c \in (a, b)$ with $f(c) = 0$. That is, $c^2 = K$. [5]

13. If $f, g : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, prove that the product $f(x)g(x)$ is differentiable at c. [8]

Soln. We have $\frac{f(x)g(x)-f(c)g(c)}{x-c} = \frac{f(x)g(x)-f(x)g(c)}{x-c} + \frac{f(x)g(c)-f(c)g(c)}{x-c} = f(x)\,\frac{g(x)-g(c)}{x-c} + g(c)\,\frac{f(x)-f(c)}{x-c}$. By a theorem in class, f is continuous at c, that is, $\lim_{x\to c} f(x) = f(c)$. So

$$\lim_{x\to c} \frac{f(x)g(x) - f(c)g(c)}{x - c} = \lim_{x\to c} f(x)\,\frac{g(x) - g(c)}{x - c} + g(c)\,\frac{f(x) - f(c)}{x - c} = f(c)g'(c) + g(c)f'(c).$$

14. (a) State the mean value theorem. [6]
(b) Prove that if $f'(x) = g'(x)$ for all $x \in (a, b)$ then $f(x) = g(x) + C$ on (a, b) for a constant C. [4]

Soln. (a) If $f : [a, b] \to \mathbb{R}$ is continuous, and f is differentiable on (a, b), then there exists $c \in (a, b)$ with $f'(c) = (f(b) - f(a))/(b - a)$. [6]

(b) Let $h = f - g$, then $h' = 0$, so by another corollary of the MVT, $h = C$ for a constant C. Thus $f(x) = g(x) + C$. [4]

15. (a) State Riemann's condition for integrability. [4]
(b) What is a uniformly continuous function? [4]
(c) Prove that a continuous function $f : [a, b] \to \mathbb{R}$ is integrable. [15]

Soln. (a) Riemann's condition states that a bounded function $f : [a, b] \to \mathbb{R}$ is integrable iff for every $\epsilon > 0$ there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. [4]

(b) $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in D$, and $|x - y| < \delta$. [4]

(c) By a theorem in class, f is uniformly continuous since $[a, b]$ is compact. Thus given $\epsilon > 0$, there is a number $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{b-a}$ whenever $x, y \in [a, b]$ and $|x - y| < \delta$. Choose a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ such that $\Delta x_k = x_k - x_{k-1} < \delta$ for every $k = 1, 2, \cdots, n$. Consider the interval $[x_{k-1}, x_k]$. By the Min-max theorem, f has a maximum value M_k and a minimum value m_k on this interval; so there are numbers s and t in $[x_{k-1}, x_k]$ with $f(s) = M_k, f(t) = m_k$. Since $|s - t| \leq \Delta x_k < \delta$, we conclude that

$$M_k - m_k = |f(s) - f(t)| < \frac{\epsilon}{b-a} .$$

Now

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} \Delta x_k \, M_k - \sum_{k=1}^{n} \Delta x_k \, m_k = \sum_{k=1}^{n} \Delta x_k \, (M_k - m_k),$$

and so

$$U(f, P) - L(f, P) < \sum_{k=1}^{n} \Delta x_k \frac{\epsilon}{b-a} = (b-a) \frac{\epsilon}{b-a} = \epsilon.$$

Thus f satisfies 'Riemann condition' (b) above, and so f is integrable. [15]

16. State and prove the second fundamental theorem of calculus (about integrating a derivative). [5+18]

Soln. If $f : [a, b] \to \mathbb{R}$ is continuous, and is differentiable on (a, b), and if f' is integrable on $[a, b]$ (set $f'(a) = f'(b) = 0$ if they are not already defined), then $\int_a^b f' \, dx = f(b) - f(a)$.

Proof: Suppose that $P = \{x_0, x_1, \cdots, x_n\}$ is a partition of $[a, b]$. By the MVT on $[x_{k-1}, x_k]$ there is a number $t_k \in (x_{k-1}, x_k)$ such that $f(x_k) - f(x_{k-1}) = f'(t_k)(x_k - x_{k-1})$. Thus

$$f(b) - f(a) = \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) = \sum_{k=1}^{n} f'(t_k)(x_k - x_{k-1}).$$

On the other hand, we have by an observation in class about Riemann sums,

$$L(f', P) \leq \sum_{k=1}^{n} f'(t_k)(x_k - x_{k-1}) \leq U(f', P),$$

and so

$$L(f', P) \leq f(b) - f(a) \leq U(f', P).$$

Taking the supremum over partitions P we get

$$\int_a^b f' \, dx = L(f') = \sup\{L(f', P) : \text{partitions } P\} \leq f(b) - f(a).$$

Similarly, taking the infimum over partitions P we get

$$f(b) - f(a) \leq U(f') = \int_a^b f' \, dx.$$

Thus $\int_a^b f' \, dx = f(b) - f(a)$.