NAME: ____________________________ Student #: ________________________

Department of Mathematics, University of Houston
Math 3333 - Intermediate Analysis - David Blecher
Test 2 - Mock Exam

Instructions. Show all working and reasoning, the points are almost all for logical, complete reasoning. Time = 90 mins. [Approximate point values]

1. (a) What does it mean for a sequence (s_n) of real numbers to converge to a number s? [5]
   (b) Prove that if (s_n) and (t_n) are convergent sequences, then $(s_n t_n)$ is a convergent sequence. [15].

2. Prove that a decreasing bounded sequence (a_n) converges to $\inf_n a_n$.

3. (a) What is the definition of a Cauchy sequence?
   (b) Suppose that (s_n) is a sequence with $|s_{n+1} - s_n| \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$. Show that (s_n) is a Cauchy sequence.
   (c) Is the sequence in (b) convergent? Why?

4. Here $f : D \to \mathbb{R}$, and c is an accumulation point of D. Mark each statement True or False. If it is true, give a simple reason. If it is false, give a counterexample (you don't need to show that it is a counterexample).
   (a) Every sequence of real numbers has a convergent subsequence.
   (b) If $\lim_{x\to c} f(x) \neq L$ then there is a sequence (s_n) in D which converges to c, but $(f(s_n))$ does not converge to L.
   (c) If $f : D \to \mathbb{R}$ is continuous and bounded on D, then $f(x)$ has a maximum and a minimum value on D.

5. Suppose that $f : (a, b) \to \mathbb{R}$, $g : (a, b) \to \mathbb{R}$, $L \in \mathbb{R}$, and $a < c < b$.
   (a) Prove that $\lim_{x\to c} f(x) = L$ iff whenever (s_n) is a sequence in $(a, b)\setminus\{c\}$ with $\lim_n s_n = c$, then $\lim_n f(s_n) = L$.
   (b) Prove that if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then $\lim_{x\to c} f(x)g(x) = LM$.
   (c) Using (a) show that if $g(x) \leq f(x) \leq h(x)$ for all $x \in (a, b)$, and if $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} f(x) = L$.
   (d) Show that if f is continuous, and $f(r) = 0$ for all rational numbers $r \in (a, b)$, then $f(x) = 0$ for all $x \in (a, b)$.
6. (a) Give the $\epsilon - \delta$ definition for a function $f : (a, b) \to \mathbb{R}$ to be continuous at a point $c \in (a, b)$.
   (b) List as many other conditions as you know that are equivalent to $f : (a, b) \to \mathbb{R}$ being continuous at $c \in (a, b)$.
   (c) Using the $\epsilon - \delta$ definition, show that the function $x^3 - x$ is continuous at $x = -1$.

7. State and prove the 'min-max theorem'.