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Math 3333 - Intermediate Analysis - David Blecher
Mock exam Test 2 - Solutions.

1. (a) What does it mean for a sequence (s_n) of real numbers to converge? [5]
   (b) Prove that if (s_n) and (t_n) are convergent sequences, then (s_nt_n) is a convergent sequence. [15].

   Solutions: (a) That given any $\epsilon > 0$, $\exists N$ such that $|s_n - s| < \epsilon$ whenever $n \geq N$.

   (b) We use Fact 6 from Handout on sequences: Note that

$$|s_nt_n - st| = |s_nt_n - s_nt + s_nt - st| \leq |s_nt_n - s_nt| + |s_nt - st| = |s_n(t_n - t)| + |t(s_n - s)| = |s_n||t_n - t| + |t||s_n - s|.$$

Now $|s_n - s| \to 0$, so $|t||s_n - s| \to 0$, as $n \to \infty$, by Fact 3. On the other hand, since (s_n) is convergent, it is bounded, by Fact 1. Thus $(|s_n|)$ is bounded. By the final assertion of Fact 3, $|s_n||t_n - t| \to 0$ as $n \to \infty$. By the first assertion of Fact 3, we now see that $|s_n||t_n - t| + |t||s_n - s| \to 0$ as $n \to \infty$. Since $|s_nt_n - st| \leq |s_n||t_n - t| + |t||s_n - s|$, by Fact 6 we deduce that $s_nt_n \to st$ as $n \to \infty$.

2. Prove that a decreasing bounded sequence (a_n) converges to $\inf_n a_n$.

   Solution: Let $\alpha = \inf_n a_n$. If $\epsilon > 0$ then $\alpha + \epsilon$ is not a lower bound for the a_n's, so there exists an N such that $a_N < \alpha + \epsilon$. If $n \geq N$ then

$$\alpha - \epsilon < \alpha \leq a_n \leq a_N < \alpha + \epsilon,$$

and so $|a_n - \alpha| < \epsilon$ for $n \geq N$. Thus $a_n \to \alpha$.

3. (a) What is the definition of a Cauchy sequence?
   (b) Suppose that (s_n) is a sequence with $|s_{n+1} - s_n| \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$. Show that (s_n) is a Cauchy sequence.
   (c) Is the sequence in (b) convergent? Why?

   Solutions (a) That given any $\epsilon > 0$, $\exists N$ such that $|s_n - s_m| < \epsilon$ whenever $m \geq n \geq N$.

   (b)

$$|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} - s_{m-2} + \cdots + s_{n+1} - s_n| \leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \cdots + |s_{n+1} - s_n|$$
$$\leq \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \cdots + \frac{1}{2^n} \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots .$$

The latter is a geometric series with sum $\frac{2}{2^n} = \frac{1}{2^{n-1}}$. If $\epsilon > 0$ is given choose N with $\frac{1}{2^{N-1}} = \epsilon$ (so $N = 1 + \log_2(1/\epsilon)$). If $m \geq n \geq N$ then as above

$$|s_m - s_n| \leq \frac{1}{2^{n-1}} \leq \frac{1}{2^{N-1}} = \epsilon,$$

That is, (s_n) is a Cauchy sequence.

   (c) Yes, since it is Cauchy, and we proved in class that every Cauchy sequence is convergent.

4. Here $f : D \to \mathbb{R}$, and c is an accumulation point of D. Mark each statement True or False. If it is true, give a simple reason. If it is false, give a counterexample (you don't need to show that it is a counterexample).
   (a) Every sequence of real numbers has a convergent subsequence.
   (b) If $\lim_{x\to c} f(x) \neq L$ then there is a sequence (s_n) in D which converges to c, but $(f(s_n))$ does not converge to L.
   (c) If $f : D \to \mathbb{R}$ is continuous and bounded on D, then $f(x)$ has a maximum and a minimum value on D.

Solutions (a) False, consider the sequence $1, 2, 3, \cdots$.

(b) True, by a 'Consequence' in the classnotes after the main theorem in Section 20.

(c) False. Let $f(x) = x$ on $D = (0, 1)$.

5. Suppose that $f : (a, b) \to \mathbb{R}$, $g : (a, b) \to \mathbb{R}$, $L \in \mathbb{R}$, and $a < c < b$.
   (a) Prove that $\lim_{x\to c} f(x) = L$ iff whenever (s_n) is a sequence in $(a, b)\setminus\{c\}$ with $\lim_n s_n = c$, then $\lim_n f(s_n) = L$.
   (b) Prove that if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then $\lim_{x\to c} f(x)g(x) = LM$.
   (c) Using (a) show that if $g(x) \leq f(x) \leq h(x)$ for all $x \in (a, b)$, and if $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} f(x) = L$.
   (d) Show that if f is continuous, and $f(r) = 0$ for all rational numbers $r \in (a, b)$, then $f(x) = 0$ for all $x \in (a, b)$.

Solutions (a) See Classnotes or Text (this is the main theorem in Section 20).

(b) See Classnotes or Text.

(c) Let $s_n \in (a, b)$ with $s_n \neq c$ and $s_n \to c$. Then $g(s_n) \leq f(s_n) \leq h(s_n)$. By (a) used twice, we have $g(s_n) \to L$ and $h(s_n) \to L$. By squeezing (Fact 5 for sequences), $f(s_n) \to L$. By (a) again, $\lim_{x\to c} f(x) = L$.

(d) If $x \in (a, b)$, and $n \in \mathbb{N}$, by the density of the rationals we may choose a rational number $r_n \in (a, b)$ with $|x - r_n| < 1/n$. Then $r_n \to r$ by Fact 6 (Sequences), so by (a) above $f(r_n) \to f(x)$. But $f(r_n) = 0$ so that $f(x) = 0$.

6. (a) Give the $\epsilon - \delta$ definition for a function $f : (a, b) \to \mathbb{R}$ to be continuous at a point $c \in (a, b)$.
   (b) List as many other conditions as you know that are equivalent to $f : (a, b) \to \mathbb{R}$ being continuous at $c \in (a, b)$.
   (c) Using the $\epsilon - \delta$ definition, show that the function $x^3 - x$ is continuous at $x = -1$.

Solutions (a) See classnotes, or Text.

(b) Here are 3 conditions: Firstly: $f(s_n) \to f(c)$ whenever $s_n \to c$, $s_n \in (a, b)$. Secondly: $\lim_{x\to c} f(x) = f(c)$. Thirdly: For any neighborhood V of $f(c)$ there exists a neighborhood U of c such that $f(x) \in V$ for all $x \in U \cap (a, b)$.

(c) Let $f(x) = x^3 - x$, then $f(-1) = -1 + 1 = 0$. So

$$|f(x) - f(0)| = |x^3 - x - 0| = |x(x-1)(x+1)| = |x||x-1||x+1|.$$

If $|x - 1| < 1$ then $0 < x < 2$ and $1 < x + 1 < 3$ so that $|x||x + 1| < 2.3 = 6$. Choose $\delta > 0$ so that $\delta < 1$ and $\delta < \epsilon/6$. If $|x - 1| < \delta$ then by the above we have

$$|f(x) - f(0)| = |x||x-1||x+1| < 6|x-1| < 6\delta < \epsilon.$$

Thus we have verified the $\epsilon - \delta$ definition for continuity at -1.

7. State and prove the 'min-max theorem'.

Solution: See classnotes, or Text Corollary 22.3.