NAME: ______________________

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Math 3333 - Intermediate Analysis - David Blecher
Test 3—August 2010

KEY

Instructions. Show all working and reasoning, the points are almost all for logical, complete reasoning. If you use a result from the class notes, state it, but you need not prove it unless you are asked to. [Approximate point values in parentheses, total = 100 points, but there are 5 bonus points]

1. If f is integrable on $[a, b]$ then does it follow that f is differentiable on (a, b)? Prove it or give a counterexample (and short explanation why your example fits the requirements). [7]

Solution. No. For example $|x|$ on $[-1, 1]$ is not differentiable at 0, but is continuous, so integrable by a theorem from class.

2. If $f(x)$ is differentiable at c, prove that $f(x)$ is continuous at c. [7]

Solution. $f(x) = \frac{f(x)-f(c)}{x-c}(x - c) + f(c) \to f'(c)0 + f(c) = f(c)$ as $x \to c$.

3. Let $f(x) = x^2|x|$. Prove that f is differentiable at every point. [9]

Solution. $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2|x|}{x} = \lim_{x\to 0} x|x| = 0$, by squeezing, since $|x|x|| = x^2 \to 0$ as $x \to 0$. So f is differentiable at 0. If $c > 0$ then $f(x) = x^3$ near c, and this is differentiable (by a result from class). Similarly, if $c < 0$ then $f(x) = -x^3$ near c, and this is differentiable.

4. (a) State and prove Rolle's theorem. [6+9]
(b) State the mean value theorem. [6]
(c) Prove that if $f'(x) \leq 0$ for all $x \in (a, b)$ then $f(x)$ is decreasing on (a, b). [7]

Solution. (a) Rolle's theorem states: If $f : [a, b] \to \mathbb{R}$ is continuous, and f is differentiable on (a, b), and $f(a) = f(b) = 0$, then there exists $c \in (a, b)$ with $f'(c) = 0$.

To prove this, note that it is clearly true if f is constant. By the MAX-MIN theorem f has a maximum and a minimum value on $[a, b]$. One of these values must be nonzero if f is not constant, and hence it is achieved at a point c which is not a or b. By the first derivative test, $f'(c) = 0$.

(b) If $f : [a, b] \to \mathbb{R}$ is continuous, and f is differentiable on (a, b), then there exists $c \in (a, b)$ with $f'(c) = (f(b) - f(a))/(b - a)$.

(c) If $a < x < y < b$ then by the MVT there exists c such that $f(y) - f(x) = f'(c)(x - y) \leq 0$, since $f'(c) \leq 0$. Thus $f(x) \leq f(y)$. So f is decreasing on (a, b).

5. (a) What can you say about a function $f:(a,b)\to\mathbb{R}$ which is one-to-one and continuous? List as many consequences as you know. [6]

(b) State the inverse function theorem. ~~[4]~~ [10]

Solution. (a) f is either strictly increasing or strictly decreasing; its range is an open interval, f^{-1} is also continuous.

(b) Suppose that $f:(a,b)\to\mathbb{R}$ is a differentiable function with $f'(x)\neq 0$ for all $x\in(a,b)$. Then f is one-to-one, its range $f((a,b))$ is an open interval (c,d), and for any $y\in(c,d)$, we have $(f^{-1})'(y)=\frac{1}{f'(x)}$, where $f(x)=y$.

6. (a) Define a partition P of $[a,b]$, define $L(f,P)$ and the lower integral $L(f)$. [10]

(b) What does it mean to say that a bounded function $f:[a,b]\to\mathbb{R}$ is integrable? [2]

(c) Give an example of a bounded function on $[0,1]$ that is not integrable (you need not explain why). [3]

7(a) ~~(d)~~ If f and g are integrable on $[a,b]$, and $f\le g$ on $[a,b]$, prove that $\int_a^b f\,dx\le\int_a^b g\,dx$. ~~[8]~~ [9]

Solution. (a) (a) A partition P is a set $\{x_0,x_1,\cdots,x_n\}$ where $a=x_0<x_1<x_2<\cdots<x_n=b$.

Define $L(f,P)=\sum_{k=1}^n m_k(x_k-x_{k-1})$, where $m_k=\inf\{f(t):x_{k-1}\le t\le x_k\}$. Define $L(f)=\sup\{L(f,P)\}$, the supremum over all partitions P of $[a,b]$.

(b) It means that $L(f)=U(f)$. 2

(c) $f(x)=1$ for rational x, $f(x)=0$ for irrational x. 3

7(a) ~~(d)~~ If we take a partition P of $[a,b]$, then

$$m_k=\inf\{f(x):x\in[x_{k-1},x_k]\}\le\inf\{g(x):x\in[x_{k-1},x_k]\}.$$

Call this last number m_k'. Then

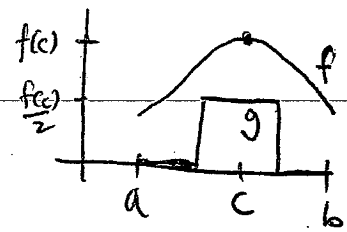
$$L(f,P)=\sum_{k=1}^n m_k\,\Delta x_k\le\sum_{k=1}^n m_k'\,\Delta x_k=L(g,P)\le L(g)=\int_a^b g\,dx.$$

Taking the supremum over all partitions P we deduce that $L(f)\le\int_a^b g\,dx$, which is what we need since $L(f)=\int_a^b f\,dx$.

7(b) ~~8~~. Suppose that $f(x)\ge 0$ for all $x\in[a,b]$, and that f is continuous on $[a,b]$, and $\int_a^b f\,dx=0$. Prove that f is always 0 on $[a,b]$. [14]

Solution. Suppose that $f(c)>0$ for some $c\in(a,b)$, we will get a contradiction. By an earlier question in the section on limits and continuity, there is a neighborhood U of c in $[a,b]$ such that $f(x)>f(c)/2$ for all $x\in U$. Let g be the function on $[a,b]$ which is 0 everywhere, except on a closed interval centered at c inside U, where $g(x)=f(c)/2$. Clearly $\int_a^b g\,dx>0$. Then $g\le f$ so by (a) we have $\int_a^b f\,dx\ge\int_a^b g\,dx>0$. This contradicts the fact that $\int_a^b f\,dx=0$.

Picture:



Alternative solution: If U above is (h,k), let $P=\{a,h,k,b\}$. Then

$$\int_a^b f\,dx\ge L(f,P)=(h-a)m_1+(k-h)m_2+(b-k)m_3$$
$$\ge 0+(k-h)\frac{f(c)}{2}+0 \quad \text{(see picture)}$$
$$>0.\ \text{a contradiction}$$