1.8 (a) Seven is not prime and $2 + 2 \neq 4$.
(b) M is bounded and M is not compact.
(c) Roses are red and violets are blue, but I do not love you.

EXERCISES

*Exercises marked with * are used in later sections and exercises marked with ☆ have hints or solutions in the back of the book.*

1.1 Mark each statement True or False. Justify each answer.
(a) In order to be classified as a statement, a sentence must be true.
(b) Some statements are both true and false.
(c) When statement p is true, then its negation $\sim p$ is false.
(d) A statement and its negation may both be false.
(e) In mathematical logic, the word "or" has an inclusive meaning.

1.2 Mark each statement True or False. Justify each answer.
(a) In an implication $p \Rightarrow q$, statement p is referred to as the proposition.
(b) The only case where $p \Rightarrow q$ is false is when p is true and q is false.
(c) "If p, then q" is equivalent to "p whenever q."
(d) The negation of a conjunction is the disjunction of the negations of the individual parts.
(e) The negation of $p \Rightarrow q$ is $q \Rightarrow p$.

1.3 Write the negation of each statement. ☆
(a) M is a cyclic subgroup.
(b) The interval $[0,3]$ is finite.
(c) The relation R is reflexive and symmetric.
(d) The set S is finite or denumerable.
(e) If $x > 3$, then $f(x) > 7$.
(f) If f is continuous and A is connected, then $f(A)$ is connected.
(g) If K is compact, then K is closed and bounded.

1.4 Write the negation of each statement.
(a) The relation R is transitive.
(b) The set of rational numbers is bounded.
(c) The function f is injective and surjective.
(d) $x < 5$ or $x > 7$.
(e) If x is in A, then $f(x)$ is not in B.
(f) If f is continuous, then $f(S)$ is closed and bounded.
(g) If K is closed and bounded, then K is compact.

2.2 Mark each statement True or False. Justify each answer.

(a) The symbol "$\exists$" means "there exist several."
(b) If a variable is used in the antecedent of an implication without being quantified, then the universal quantifier is assumed to apply.
(c) The order in which quantifiers are used affects the truth value.

2.3 Write the negation of each statement. ☆

(a) All the roads in Yellowstone are open.
(b) Some fish are green.
(c) No even integer is prime.
(d) $\exists\, x < 3 \ni x^2 \geq 10$.
(e) $\forall\, x$ in A, $\exists\, y < k \ni 0 < f(y) < f(x)$.
(f) If $n > N$, then $\forall\, x$ in S, $|f_n(x) - f(x)| < \varepsilon$.

2.4 Write the negation of each statement.

(a) Some basketball players at Central High are short.
(b) All of the lights are on.
(c) No bounded interval contains infinite many integers.
(d) $\exists\, x$ in $S \ni x \geq 5$.
(e) $\forall\, x \ni 0 < x < 1$, $f(x) < 2$ or $f(x) > 5$.
(f) If $x > 5$, then $\exists\, y > 0 \ni x^2 > 25 + y$.

2.5 Determine the truth value of each statement, assuming that x, y, and z are real numbers. ☆

(a) $\exists\, x \ni \forall\, y\, \exists\, z \ni x + y = z$.
(b) $\exists\, x \ni \forall\, y$ and $\forall\, z$, $x + y = z$.
(c) $\forall\, x$ and $\forall\, y$, $\exists\, z \ni y - z = x$.
(d) $\forall\, x$ and $\forall\, y$, $\exists\, z \ni xz = y$.
(e) $\exists\, x \ni \forall\, y$ and $\forall\, z$, $z > y$ implies that $z > x + y$.
(f) $\forall\, x$, $\exists\, y$ and $\exists\, z \ni z > y$ implies that $z > x + y$.

2.6 Determine the truth value of each statement, assuming that x, y and z are real numbers.

(a) $\forall\, x$ and $\forall\, y$, $\exists\, z \ni x + y = z$.
(b) $\forall\, x\, \exists\, y \ni \forall\, z$, $x + y = z$.
(c) $\exists\, x \ni \forall\, y$, $\exists\, z \ni xz = y$.
(d) $\forall\, x$ and $\forall\, y$, $\exists\, z \ni yz = x$.
(e) $\forall\, x\, \exists\, y \ni \forall\, z$, $z > y$ implies that $z > x + y$.
(f) $\forall\, x$ and $\forall\, y$, $\exists\, z \ni z > y$ implies that $z > x + y$.

2.7 Below are two strategies for determining the truth value of a statement involving a positive number x and another statement $P(x)$.

(*i*) Find some $x > 0$ such that $P(x)$ is true.
(*ii*) Let x be the name for any number greater than 0 and show $P(x)$ is true.

For each statement below, indicate which strategy is more appropriate.

(a) $\forall\, x > 0,\, P(x)$. ☆
(b) $\exists\, x > 0 \ni P(x)$. ☆
(c) $\exists\, x > 0 \ni \sim P(x)$.
(d) $\forall\, x > 0, \sim P(x)$.

2.8 Which of the following best identifies f as a constant function, where x and y are real numbers.

(a) $\exists\, x \ni \forall\, y,\, f(x) = y$.
(b) $\forall\, x\, \exists\, y \ni f(x) = y$.
(c) $\exists\, y \ni \forall\, x,\, f(x) = y$.
(d) $\forall\, y\, \exists\, x \ni f(x) = y$.

2.9 Determine the truth value of each statement, assuming x is a real number. ☆

(a) $\exists\, x \in [2,4] \ni x < 7$.
(b) $\forall\, x \in [2,4],\, x < 7$.
(c) $\exists\, x \ni x^2 = 5$.
(d) $\forall\, x,\, x^2 = 5$.
(e) $\exists\, x \ni x^2 \neq -3$.
(f) $\forall\, x,\, x^2 \neq -3$.
(g) $\exists\, x \ni x \div x = 1$
(h) $\forall\, x,\, x \div x = 1$.

2.10 Determine the truth value of each statement, assuming x is a real number.

(a) $\exists\, x \in [3,5] \ni x \geq 4$.
(b) $\forall\, x \in [3,5],\, x \geq 4$
(c) $\exists\, x \ni x^2 \neq 3$.
(d) $\forall\, x,\, x^2 \neq 3$.
(e) $\exists\, x \ni x^2 = -5$.
(f) $\forall\, x,\, x^2 = -5$.
(g) $\exists\, x \ni x - x = 0$.
(h) $\forall\, x,\, x - x = 0$.

Exercises 2.11 to 2.19 give certain properties of functions that we shall encounter later in the text. You are to do two things: (a) rewrite the defining conditions in logical symbolism using $\forall$, $\exists$, $\ni$, and $\Rightarrow$, as appropriate; and (b) write the negation of part (a) using the same symbolism. It is not necessary that you understand precisely what each term means.

Example: A function f is odd iff for every x, $f(-x) = -f(x)$.

(a) defining condition: $\forall\, x,\, f(-x) = -f(x)$.

(b) negation: $\exists\, x \ni f(-x) \neq -f(x)$.

2.11 A function f is *even* iff for every x, $f(-x) = f(x)$. ☆

2.12 A function f is *periodic* iff there exists a $k > 0$ such that for every x, $f(x + k) = f(x)$.

2.13 A function f is *increasing* iff for every x and for every y, if $x \leq y$, then $f(x) \leq f(y)$. ☆

2.14 A function f is *strictly decreasing* iff for every x and for every y, if $x < y$, then $f(x) > f(y)$.

2.15 A function $f: A \to B$ is *injective* iff for every x and y in A, if $f(x) = f(y)$, then $x = y$. ☆

2.16 A function $f: A \to B$ is *surjective* iff for every y in B there exists an x in A such that $f(x) = y$.

2.17 A function $f: D \to R$ is *continuous* at $c \in D$ iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$ and $x \in D$. ☆

2.18 A function f is *uniformly continuous on a set* S iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever x and y are in S and $|x - y| < \delta$.

2.19 The real number L is the *limit* of the function $f: D \to R$ at the point c iff for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in D$ and $0 < |x - c| < \delta$. ☆

2.20 Consider the following sentences:

(a) The nucleus of a carbon atom consists of protons and neutrons.
(b) Jesus Christ rose from the dead and is alive today.
(c) Every differentiable function is continuous.

Each of these sentences has been affirmed by some people at some time as being "true." Write an essay on the nature of truth, comparing and contrasting its meaning in these (and possibly other) contexts. You might also want to consider some of the following questions: To what extent is truth absolute? To what extent can truth change with time? To what extent is truth based on opinion? To what extent are people free to accept as true anything they wish?

Section 3 TECHNIQUES OF PROOF: I

In the first two sections we introduced some of the vocabulary of logic and mathematics. Our aim is to be able to read and write mathematics, and this requires more than just vocabulary. It also requires syntax. That is, we need to understand how statements are combined to form the mysterious mathematical entity known as a proof. Since this topic tends to be intimidating to many students, let us ease into it gently by first considering the two main types of logical reasoning: inductive reasoning and deductive reasoning.

(c) The inverse of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$.
(d) To prove "$\forall\, n,\ p(n)$" is true, it takes only one example.
(e) To prove "$\exists\, n \ni p(n)$" is true, it takes only one example.

3.2 Mark each statement True or False. Justify each answer.

(a) When an implication $p \Rightarrow q$ is used as a theorem, we refer to q as the conclusion.
(b) A statement that is always false is called a lie.
(c) The converse of $p \Rightarrow q$ is $q \Rightarrow p$.
(d) To prove "$\forall\, n,\ p(n)$" is false, it takes only one counterexample.
(e) To prove "$\exists\, n \ni p(n)$" is false, it takes only one counterexample.

3.3 Write the contrapositive of each implication. ☆

(a) If all roses are red, then all violets are blue.
(b) H is normal if H is not regular.
(c) If K is closed and bounded, then K is compact.

3.4 Write the converse of each implication in Exercise 3.3.

3.5 Write the inverse of each implication in Exercise 3.3.

3.6 Provide a counterexample for each statement.

(a) For every real number x, if $x^2 > 4$ then $x > 2$.
(b) For every positive integer n, $n^2 + n + 41$ is prime.
(c) Every triangle is a right triangle.
(d) No integer greater than 100 is prime.
(e) Every prime is an odd number.
(f) For every positive integer n, $3n$ is divisible by 6.
(g) If x and y are unequal positive integers and xy is a perfect square, then x and y are perfect squares.
(h) Every real number has a reciprocal.
(i) For all real numbers $x > 0$, we have $x^2 \le x^3$.
(j) The reciprocal of a real number $x \ge 1$ is a real number y such that $0 < y < 1$.
(k) $3^n + 2$ is prime for all $n \in \mathbb{N}$.
(l) No rational number satisfies the equation $x^3 + (x-1)^2 = x^2 + 1$.
(m) No rational number satisfies the equation $x^4 + (1/x) - \sqrt{x+1} = 0$.

3.7 Suppose p and q are integers. Recall that an integer m is even iff $m = 2k$ for some integer k and m is odd iff $m = 2k + 1$ for some integer k. Prove the following. [You may use the fact that the sum and product of integers is again an integer.]

(a) If p is odd and q is odd, then $p + q$ is even.
(b) If p is odd and q is odd, then pq is odd.
(c) If p is odd and q is even, then $p + q$ is odd.
(d) If p is even and q is even, then $p + q$ is even.
(e) If p is even or q is even, then pq is even.

(f) If pq is odd, then p is odd and q is odd.
(g) If p^2 is even, then p is even. ☆
(h) If p^2 is odd, then p is odd.

3.8 Let f be the function given by $f(x) = 3x - 5$. Use the contrapositive implication to prove: If $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

3.9 In each part, a list of hypotheses is given. These hypotheses are assumed to be true. Using tautologies from Example 3.12, you are to establish the desired conclusion. Indicate which tautology you are using to justify each step. ☆

(a) Hypotheses: $r \Rightarrow \sim s,\ t \Rightarrow s$
Conclusion: $r \Rightarrow \sim t$
(b) Hypotheses: $r, \sim t, (r \wedge s) \Rightarrow t$
Conclusion: $\sim s$
(c) Hypotheses: $r \Rightarrow \sim s, \sim r \Rightarrow \sim t, \sim t \Rightarrow u, v \Rightarrow s$
Conclusion: $\sim v \vee u$

3.10 Repeat Exercise 3.9 for the following hypotheses and conclusions.

(a) Hypotheses: $\sim r, (\sim r \wedge s) \Rightarrow r$
Conclusion: $\sim s$
(b) Hypotheses: $\sim t, (r \vee s) \Rightarrow t$
Conclusion: $\sim s$
(c) Hypotheses: $r \Rightarrow \sim s, t \Rightarrow u, s \vee t$
Conclusion: $\sim r \vee u$

3.11 Assume that the following two hypotheses are true: (1) If the basketball center is healthy or the point guard is hot, then the team will win and the fans will be happy; and (2) if the fans are happy or the coach is a millionaire, then the college will balance the budget. Derive the following conclusion: If the basketball center is healthy, then the college will balance the budget. Using letters to represent the simple statements, write out a formal proof in the format of Exercise 3.9.

Section 4 TECHNIQUES OF PROOF: II

Mathematical theorems and proofs do not occur in isolation, but always in the context of some mathematical system. For example, in Section 3 when we discussed a conjecture related to prime numbers, the natural context of that discussion was the positive integers. In Example 3.7 when talking about odd and even numbers, the context was the set of all integers. Very often a theorem will make no explicit reference to the mathematical system in which it is being proved; it must be inferred from the context. Usually, this causes

EXERCISES

*Exercises marked with * are used in later sections and exercises marked with ☆ have hints or solutions in the back of the book.*

4.1 Mark each statement True or False. Justify each answer.

(a) To prove a universal statement $\forall\, x,\ p(x)$, we let x represent an arbitrary member from the system under consideration and show that $p(x)$ is true.

(b) To prove an existential statement $\exists\, x \ni p(x)$, we must find a particular x in the system for which $p(x)$ is true.

(c) In writing a proof, it is important to include all the logical steps.

4.2 Mark each statement True or False. Justify each answer.

(a) A proof by contradiction may use the tautology $(\sim p \Rightarrow c) \Leftrightarrow p$.

(b) A proof by contradiction may use the tautology $[(p \vee \sim q) \Rightarrow c] \Leftrightarrow (p \Rightarrow q)$.

(c) Definitions often play an important role in proofs.

4.3 Prove: There exists an integer n such that $n^2 + 3n/2 = 1$. Is this integer unique? ☆

4.4 Prove: There exists a rational number x such that $x^2 + 3x/2 = 1$. Is this rational number unique?

4.5 Prove: For every real number $x > 3$, there exists a real number $y < 0$ such that $x = 3y/(2+y)$. ☆

4.6 Prove: For every real number $x > 1$, there exist two distinct positive real numbers y and z such that

$$x = \frac{y^2+9}{6y} = \frac{z^2+9}{6z}.$$

4.7 Prove: If $x^2 + x - 6 \geq 0$, then $x \leq -3$ or $x \geq 2$. ☆

4.8 Prove: If $x/(x-1) \leq 2$, then $x < 1$ or $x \geq 2$.

4.9 Prove: $\log_2 7$ is irrational. ☆

4.10 Prove: If x is a real number, then $|x-2| \leq 3$ implies that $-1 \leq x \leq 5$.

4.11 Consider the following theorem: "If m^2 is odd, then m is odd." Indicate what, if anything, is wrong with each of the following "proofs."

(a) Suppose m is odd. Then $m = 2k + 1$ for some integer k. Thus $m^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd. Thus if m^2 is odd, then m is odd.

(b) Suppose m is not odd. Then m is even and $m = 2k$ for some integer k. Thus $m^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is even. Thus if m is not odd, then m^2 is not odd. It follows that if m^2 is odd, then m is odd.

4.12 Consider the following theorem: "If $xy = 0$, then $x = 0$ or $y = 0$." Indicate what, if anything, is wrong with each of the following "proofs."

(a) Suppose $xy = 0$ and $x \neq 0$. Then dividing both sides of the first equation by x we have $y = 0$. Thus if $xy = 0$, then $x = 0$ or $y = 0$.

(b) There are two cases to consider. First suppose that $x = 0$. Then $x \cdot y =$ $x \cdot 0 = 0$. Similarly, suppose that $y = 0$. Then $x \cdot y = x \cdot 0 = 0$. In either case, $x \cdot y = 0$. Thus if $xy = 0$, then $x = 0$ or $y = 0$.

4.13 Suppose x and y are real numbers. Recall that a real number m is rational iff $m = p/q$ where p and q are integers and $q \neq 0$. If a real number is not rational, then it is irrational. Prove the following. [You may use the fact that the sum and product of integers is again an integer.]

(a) If x is rational and y is rational, then $x + y$ is rational.

(b) If x is rational and y is rational, then xy is rational.

(c) If x is rational and y is irrational, then $x + y$ is irrational. ☆

4.14 Suppose x and y are real numbers. Prove or give a counterexample. [See the definitions in Exercise 4.13.]

(a) If x is irrational and y is irrational, then $x + y$ is irrational.

(b) If $x + y$ is irrational, then x is irrational or y is irrational.

(c) If x is irrational and y is irrational, then xy is irrational.

(d) If xy is irrational, then x is irrational or y is irrational.

4.15 Consider the following theorem and proof.

Theorem: If x is rational and y is irrational, then xy is irrational.

Proof: Suppose x is rational and y is irrational. If xy is rational then we have $x = p/q$ and $xy = m/n$ for some integers p, q, m and n, with $q \neq 0$ and $n \neq 0$. It follows that

$$y = \frac{xy}{x} = \frac{m/n}{p/q} = \frac{mq}{np}.$$

This implies that y is rational, a contradiction. We conclude that xy must be rational. ♦

(a) Find a specific counterexample to show that the theorem is false.

(b) Explain what is wrong with the proof.

(c) What additional condition on x in the hypothesis would make the conclusion true?

4.16 Prove or give a counterexample: If x is irrational, then $\sqrt{x}$ is irrational.

4.17 Prove or give a counterexample: There do not exist three consecutive even integers a, b, and c such that $a^2 + b^2 = c^2$. ☆

4.18 Consider the following theorem: There do not exist three consecutive odd integers a, b, and c such that $a^2 + b^2 = c^2$.

(a) Complete the following restatement of the theorem: For every three consecutive odd integers a, b, and c, __________.

(b) Change the sentence in part (a) into an implication $p \Rightarrow q$: If a, b, and c are consecutive odd integers, then ____________.

(c) Fill in the blanks in the following proof of the theorem.

Proof: Let a, b, and c be consecutive odd integers. Then $a = 2k+1$, $b =$ __________, and $c = 2k+5$ for some integer k. Suppose $a^2 + b^2 = c^2$. Then $(2k+1)^2 + ($_________$)^2 = (2k+5)^2$.

It follows that $8k^2 + 16k + 10 = 4k^2 + 20k + 25$ and $4k^2 - 4k -$ _____ $= 0$. Thus $k = 5/2$ or $k =$ _____. This contradicts k being an ______________. Therefore, there do not exist three consecutive odd integers a, b, and c such that $a^2 + b^2 = c^2$. ◆

(d) Which of the tautologies in Example 3.12 best describes the structure of the proof?

4.19 Prove or give a counterexample: The sum of any five consecutive integers is divisible by five.

4.20 Prove or give a counterexample: The sum of any four consecutive integers is never divisible by four.

4.21 Prove or give a counterexample: For every positive integer n, $n^2 + 3n + 8$ is even. ☆

4.22 Prove or give a counterexample: For every positive integer n, $n^2 + 4n + 8$ is even.

4.23 Prove or give a counterexample: there do not exist irrational numbers x and y such that x^y is rational. ☆

4.24 Prove or give a counterexample: there do not exist rational numbers x and y such that x^y is a positive integer and y^x is a negative integer.

4.25 Prove or give a counterexample: for all $x > 0$ we have $x^2 + 1 < (x+1)^2 \leq 2(x^2 + 1)$.