NAME: __________ KEY __________

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Math 3333 - Intermediate Analysis - David Blecher
Test 1– October 11 2007.

Instructions. Time= 80. Put all books and papers at the side of the room. You only need to prove things you are asked to prove. Show all working and reasoning, the points are almost all for logical, complete reasoning. [Approximate point values are given, total = 100 + bonus].

1. Write the negation of the statement: $\forall \epsilon > 0$, $\exists N \geq 1$ s.t. $|s_n - L| < \epsilon$ whenever $n \geq N$. [6]

   Solution. $\exists \epsilon > 0$ s.t. $\forall N \geq 1$, $\exists n \geq N$ s.t. $|s_n - L| \geq \epsilon$.

2. Consider the following statement about real numbers: $\forall y$ and $\forall z$, $\exists x$ s.t. $z = xy$. Is this statement true? If so, prove it. If not, give a counterexample. [5]

   Solution. False. If $y = 0$ but $z = 1$ (or any number $\neq 0$), then there can exist no x s.t. $z = xy$.

3. Prove by mathematical induction: $1 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$. [12]

   Solution: If $n = 1$ this simply says that $1 = \frac{1}{6}6$ which is true. Assume it is true for $n = k$. If $n = k + 1$ then

$$1+2^2+3^2+\cdots+k^2+(k+1)^2 = \frac{1}{6}k(k+1)(2k+1)+(k+1)^2 = \frac{k+1}{6}(2k^2+k+6k+6) = \frac{k+1}{6}(2k^2+7k+6).$$

   On the other hand,

$$\frac{1}{6}n(n+1)(2n+1) = \frac{k+1}{6}(k+2)(2k+2+1) = \frac{k+1}{6}(2k^2+7k+6).$$

   Thus the statement is true for $n = k+1$. By mathematical induction it is true for all $n \in \mathbb{N}$.

4. (a) State the completeness axiom. [4]
   (b) If A and B are sets of real numbers which are both bounded above, prove that the set $C = \{a + b : a \in A, b \in B\}$ is also bounded above. Also, prove that $\sup C \leq \sup A + \sup B$. [5]
   (c) Prove that $\inf(-2, 0] = -2$, giving all the details. [8]

   Solution: (a) If S is a nonempty set of real numbers which is bounded above, then S has a least upper bound.
   (b) Since $a + b \leq \sup A + \sup B$ if $a \in A, b \in B$, we have that C is bounded above by $\sup A + \sup B$. Thus $\sup C \leq \sup A + \sup B$.
   (c) Certainly -2 is a lower bound of $(-2, 0]$. If $-2 < \alpha < 0$ then $-2 = \frac{-2-2}{2} < \frac{\alpha-2}{2} < \frac{\alpha+\alpha}{2} = \alpha < 0$. Thus $\frac{\alpha-2}{2} \in (-2, 0]$, and so α is not a lower bound of $(-2, 0]$. Hence -2 is the greatest lower bound of $(-2, 0]$.

5. (a) Define the term 'accumulation point' for a set S of real numbers. [5]
   (b) Is π an accumulation point of the set of rational numbers in the interval $(0, \pi)$? Explain with a complete argument. [7]
   (c) Find all the boundary points and all accumulation points of $\{1/n : n \in \mathbb{N}\}$. You do not need to prove your assertions. [4]
   (d) Is $\{1/n : n \in \mathbb{N}\}$ open? Closed? Compact? Give reasons. [8]

Solution: (a) This is a number x such that every deleted neighborhood of x contains at least one point from S.

(b) If $\epsilon > 0$ then $(\pi - \epsilon, \pi)$ contains a rational number from $(0, \pi)$, by the density of the rationals. Thus every deleted neighborhood of π contains such a rational, and so π is an accumulation point.

(c) The boundary points of $S = \{1/n : n \in \mathbb{N}\}$ are every point in S, together with 0. The only accumulation point is 0.

(d) $S = \{1/n : n \in \mathbb{N}\}$ is not open, since it contains several of its boundary points. It is not closed since it does not contain its boundary point 0. It is not compact since it is not closed.

6. (a) Define the closure $\bar{S}$ of a nonempty set S. [3]
   (b) State as many characterizations of (or ways of thinking about) $\bar{S}$ as you know. [10]
   (c) Prove that if S is a nonempty set which is bounded above, then $\sup(S) \in \bar{S}$. [11]
   (d) If S in (c) is also closed, explain why $\sup(S) \in S$. That is, S has a maximum. [4]
   (e) Show that the union of two compact sets is compact. [10]

Solution: (a) $\bar{S} = S \cup Bdy(S)$.

(b) $\bar{S} = S \cup S'$. It is also the smallest closed set containing S. It is also the set of points x which have the property that $N(x, \epsilon) \cap S \neq \emptyset$ for every $\epsilon > 0$ (or in 'English': every open interval centered at x contains at least one point from S). It is also the set of points x which are limits of sequences (s_n) where $s_n \in S \; \forall n$.

(c) $\sup(S)$ exists by the completeness axiom. Let $\alpha = \sup(S)$. We will show $\alpha \in Bdy(S) \subset \bar{S}$. If $\epsilon > 0$ is given then $(\alpha, \alpha + \epsilon)$ clearly contains points not in S. On the other hand, since $\alpha - \epsilon$ is not an upper bound for S, there exist numbers $x \in S$ with $x \in (\alpha - \epsilon, \alpha]$. Thus $N(x, \epsilon) = (\alpha - \epsilon, \alpha + \epsilon)$ contains points not in S and contains points in S. So $x \in Bdy(S)$.

(d) If S is closed it contains all its boundary points. In (c) we proved $\sup(S)$ is a boundary point. So $\sup(S) \in S$. That is, S has a maximum.

(e) Let C be a union of two compact sets A and B. Since compact sets are closed, and since a union of closed sets is closed, we see that C is closed. If K is an upper bound for A and M is an upper bound for B then $\max\{K, M\}$ is an upper bound for $A \cup B$. Similarly, $A \cup B$ has a lower bound. So C is bounded. So C is compact.