4. Sequences

4.1. Convergent sequences.

- A sequence (s_n) converges to a real number s if $\forall \epsilon > 0$, $\exists N s.t.$ $|s_n - s| < \epsilon$ $\forall n \geq N$. Saying that $|s_n - s| < \epsilon$ is the same as saying that $s - \epsilon < s_n < s + \epsilon$.
- If (s_n) converges to s then we say that s is the *limit* of (s_n) and write $s = \lim_n s_n$, or $s = \lim_{n\to\infty} s_n$, or $s_n \to s$ as $n \to \infty$, or simply $s_n \to s$.
- If (s_n) does not converge to any real number then we say that it *diverges*.
- A sequence (s_n) is called *bounded* if the set $\{s_n : n \in \mathbb{N}\}$ is a bounded set. That is, there are numbers m and M such that $m \leq s_n \leq M$ for all $n \in \mathbb{N}$. This is the same as saying that $\{s_n : n \in \mathbb{N}\} \subset [m, M]$. It is easy to see that this is equivalent to: there exists a number $K \geq 0$ such that $|s_n| \leq K$ for all $n \in \mathbb{N}$. (See the first lines of the last Section.)

Fact 1. Any convergent sequence is bounded.

Proof: Suppose that $s_n \to s$ as $n \to \infty$. Taking $\epsilon = 1$ in the definition of convergence gives that there exists a number $N \in \mathbb{N}$ such that $|s_n - s| < 1$ whenever $n \geq N$. Thus

$$|s_n| = |s_n - s + s| \leq |s_n - s| + |s| < 1 + |s|$$

whenever $n \geq N$. Now let $M = \max\{|s_1|, |s_2|, \cdots, |s_N|, 1 + |s|\}$. We have $|s_n| \leq M$ if $n = 1, 2, \cdots, N$, and $|s_n| \leq M$ if $n \geq N$. So (s_n) is bounded.

- A sequence (a_n) is called *nonnegative* if $a_n \geq 0$ for all $n \in \mathbb{N}$. To say that a nonnegative sequence converges to zero is simply to say that:

$$\forall \epsilon > 0,\ \exists N s.t.\ a_n < \epsilon \ \ \forall n \geq N.$$

Fact 2. If (s_n) is a general sequence then:

$$\lim_n s_n = s \iff \lim_n (s_n - s) = 0 \iff \lim_n |s_n - s| = 0.$$

That is, the sequence (s_n) converges to s if and only if the nonnegative sequence $(|s_n - s|)$ converges to 0.

Fact 3. If (a_n) and (b_n) are nonnegative sequences, with $\lim_n a_n = 0$ and $\lim_n b_n = 0$, and if $C \geq 0$, then

$$\lim_n a_n + b_n \ = \ \lim_n C a_n \ = \ 0.$$

Also, if $\lim_n a_n = 0$ and if (b_n) is any bounded sequence, then $\lim_n a_n b_n = 0$.

Fact 4. If (s_n) and (t_n) are sequences with $s_n \leq t_n$ for every $n \geq 1$. If $\lim_n s_n = s$ and $\lim_n t_n = t$, then $s \leq t$.

Fact 5: The 'squeezing' or 'pinching rule'. Suppose that $(s_n), (x_n)$, and (t_n) are sequences with $s_n \leq x_n \leq t_n$, for every $n \geq 1$. If $\lim_n s_n = s$ and $\lim_n t_n = s$, then $\lim_n x_n = s$.

Fact 6, sometimes call 'squeezing' too, since it is a corollary of Fact 5: Let (s_n) be a sequence, let s be a number, and suppose that $|s_n - s| \leq a_n$ for all $n \geq 1$, where (a_n) is a sequence with limit 0. Then $\lim_n s_n = s$.

Proof: We have $0 \leq |s_n - s| \leq a_n$. Now $a_n \to 0$ as $n \to \infty$, so by 'squeezing' (Fact 5), $\lim_n |s_n - s| = 0$. By Fact 2, $\lim_n s_n = s$.

Fact 7: The limit of a sequence has nothing to do with its first few terms.

Fact 8: If $\lim_n t_n = t$, and if $t \neq 0$, then there exists a number N with $|t_n| > \frac{|t|}{2}$ for all $n \geq N$.

Fact 9: If $\lim_n s_n = s$ and $\lim_n t_n = t$, then:

(1) $\lim_n s_n + t_n = s + t$;
(2) $\lim_n s_n - t_n = s - t$;
(3) $\lim_n s_n t_n = st$;
(4) $\lim_n C s_n = C s$, if C is a constant;
(5) $\lim_n \frac{s_n}{t_n} = \frac{s}{t}$, if $t \neq 0$;
(6) $\lim_n |s_n| = |s|$;
(7) $\lim_n \sqrt{t_n} = \sqrt{t}$, if $t_n \geq 0$ for all $n \in \mathbb{N}$.

Proof: By Fact 2, we have $|s_n - s| \to 0$, and $|t_n - t| \to 0$, as $n \to \infty$. To prove (1), by Fact 6 it is enough to show that $|(s_n + t_n) - (s + t)| \leq a_n$, where $a_n \to 0$ as $n \to \infty$. But

$$|(s_n + t_n) - (s + t)| = |(s_n - s) + (t_n - t)| \leq |s_n - s| + |t_n - t| \to 0,$$

using the triangle inequality, and Fact 3 in the very last step. So we have proved (1). The proof of (3) is similar, by Fact 6, it is enough to show that $|s_n t_n - st| \leq a_n$, where $a_n \to 0$, as $n \to \infty$. But

$$|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st| \leq |s_n t_n - s_n t| + |s_n t - st| = |s_n(t_n - t)| + |t(s_n - s)| = |s_n||t_n - t| + |t||s_n - s|.$$

Now $|s_n - s| \to 0$, so $|t||s_n - s| \to 0$, as $n \to \infty$, by Fact 3. On the other hand, since (s_n) is convergent, it is bounded, by Fact 1. Thus $(|s_n|)$ is bounded. By the final assertion of Fact 3, $|s_n||t_n - t| \to 0$ as $n \to \infty$. By the first assertion of Fact 3, we now see that $|s_n||t_n - t| + |t||s_n - s| \to 0$ as $n \to \infty$. Since $|s_n t_n - st| \leq |s_n||t_n - t| + |t||s_n - s|$, by Fact 6 we deduce that $s_n t_n \to st$ as $n \to \infty$. So we have proved (3).

(4) follows from (3), if we set $t_n = C$ for all n. Applying (4) with $C = -1$ shows that $\lim_n(-t_n) = -t$. Using this with (1), gives

$$\lim_n (s_n - t_n) = \lim_n (s_n + (-t_n)) = \lim_n s_n + \lim_n (-t_n) = s - t.$$

This proves (2).

For (5), we use a similar strategy to (1) and (3). Note that

$$\left|\frac{s_n}{t_n} - \frac{s}{t}\right| = \left|\frac{s_n t - t_n s}{t_n t}\right| = \frac{|s_n t - st + st - t_n s|}{|t_n||t|} \leq \frac{|s_n t - st| + |st - t_n s|}{|t_n||t|} = \frac{|s_n - s||t| + |s||t - t_n|}{|t_n||t|}.$$

By Fact 8, $\exists N$ s.t. $|t_n| > |t|/2$ for $n \geq N$. Thus for $n \geq N$,

$$\left|\frac{s_n}{t_n} - \frac{s}{t}\right| \leq \frac{|s_n - s||t| + |s||t - t_n|}{|t_n||t|} \leq \frac{2}{|t|^2}(|s_n - s||t| + |s||t - t_n|).$$

Since $|s_n - s| \to 0$ and $|t - t_n| \to 0$, as $n \to \infty$, by Fact 3 it follows that $|s_n - s||t| + |s||t - t_n| \to 0$ too. By Fact 3 again, $\frac{2}{|t|^2}(|s_n - s||t| + |s||t - t_n|) \to 0$ as $n \to \infty$. Thus we conclude from Fact 6 (in conjunction with Fact 7), that $\frac{s_n}{t_n} \to \frac{s}{t}$ as $n \to \infty$.

(6) follows by squeezing too, since we have using Theorem 3.3 (f) above, that $||s_n| - |s|| \leq |s_n - s| \to 0$. So by Fact 6, $|s_n| \to |s|$. We omit (7).

Infinite limits. If (s_n) is a sequence, then we write $\lim_n s_n = +\infty$ if $\forall M > 0$, $\exists N$ s.t. $s_n > M$ $\forall n \geq N$. We write $\lim_n s_n = -\infty$ if $\forall M > 0$, $\exists N$ s.t. $s_n < -M$ $\forall n \geq N$. Such sequences must be unbounded, and hence divergent (by the contrapositive to Fact 1).

Proposition 4.1. *Suppose that (s_n) and (t_n) are sequences such that $s_n \leq t_n$, $\forall n$.*

(a) *If $\lim_n s_n = +\infty$, then $\lim_n t_n = +\infty$.*
(b) *If $\lim_n t_n = -\infty$, then $\lim_n s_n = -\infty$.*

Theorem 4.2. *Let (s_n) be a sequence of positive numbers (so $s_n > 0, \forall n$). Then $\lim_n s_n = +\infty \Leftrightarrow \lim_n \frac{1}{s_n} = 0$.*

Theorem 4.3. *If S is a nonempty set in $\mathbb{R}$ then*

(a) *$x \in \bar{S}$ iff there is a sequence (s_n) in S with limit x.*
(b) *$x \in S'$ iff there is a sequence (s_n) in $S \setminus \{x\}$ with limit x.*
(c) *S is closed iff whenever (s_n) is a sequence in S with limit x, then $x \in S$.*

Proof. (a) ($\Leftarrow$) If (s_n) is a sequence in S with limit x, and if $\epsilon > 0$ is given, then there exists an N with $|s_n - x| < \epsilon$ if $n \geq N$. Hence $s_N \in N(x, \epsilon)$, and so $N(x, \epsilon)$ contains a point of S. Thus $x \in \bar{S}$ by Lemma 3.20.

($\Rightarrow$) If $x \in \bar{S}$, and $n \in \mathbb{N}$, then by Lemma 3.20, $N(x, \frac{1}{n})$ contains a point of S. Call this point s_n. Since $|s_n - x| < 1/n \to 0$, it follows by Fact 6 in 4.1 that $s_n \to x$.

(c) ($\Leftarrow$) Suppose that the condition in (c) about sequences holds. Let $x \in \bar{S}$. By (a), there is a sequence $s_n \to x$ with $s_n \in S$ for all $n \in \mathbb{N}$. By hypothesis, $x \in S$. So $\bar{S} \subset S$, therefore $\bar{S} = S$, and so S is closed by Corollary 3.24.

($\Rightarrow$) Suppose that S is closed, and that (s_n) is a sequence in S with $s_n \to x$. By (a), we have $x \in \bar{S} = S$ (using Corollary 3.24). $\square$

4.2. **Monotone sequences and Cauchy sequences.** Recall from Calculus II that a sequence (s_n) is *increasing* if $s_n \leq s_{n+1}$, $\forall n \in \mathbb{N}$. A sequence (s_n) is *decreasing* if $s_n \geq s_{n+1}$, $\forall n \in \mathbb{N}$. We say *strictly increasing* (resp. *strictly decreasing*) if the $\leq$ here (resp. *ge*) is replaced by $<$ (resp. $>$). A sequence (s_n) is *monotone* if it is increasing or decreasing (or both, which happens for constant sequences).

Theorem 4.4. *A monotone sequence is convergent iff it is bounded.*

Proof. ($\Rightarrow$) Every convergent sequence is bounded (by Fact 1).

($\Leftarrow$) Suppose that (s_n) is a bounded increasing sequence (the decreasing case is similar). Let $S = \{s_n : n \in \mathbb{N}\}$. This is bounded above, and let $s = \sup S$. Claim: $\lim_n \ s_n = s$. Let $\epsilon > 0$ be given. Then $s - \epsilon$ is not an upper bound for S. Thus $\exists N$ s.t. $s_N > s - \epsilon$. Hence

$$s - \epsilon < s_N \leq s_n \leq s < s + \epsilon$$

for all $n \geq N$. We've shown that $\forall \epsilon > 0\ \exists N$ s.t. $|s_n - s| < \epsilon$ for all $n \geq N$. Thus, the sequence (s_n) converges to s. □

Remark. The last proof shows that a bounded increasing (resp. decreasing) sequenc converges to its supremum (resp. infimum).

Proposition 4.5. *If* (s_n) *is an unbounded increasing (resp. decreasing) sequence, then* $\lim_n \ s_n = +\infty$ *(resp.* $= -\infty$*).*

Definition: A sequence (s_n) of real numbers is called a *Cauchy sequence* if $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ s.t. $|s_n - s_m| < \epsilon$, whenever $m \geq n \geq N$.

Lemma 4.6. *Every convergent sequence is a Cauchy sequence.*

Lemma 4.7. *Every Cauchy sequence is bounded.*

Theorem 4.8. (Cauchy test for convergence) *A sequence in* $\mathbb{R}$ *is convergent iff it is a Cauchy sequence.*

4.3. **Subsequences.** A *subsequence* of a sequence (s_n) is constructed from (s_n) by removing terms in the sequence. Associated with each subsequence is a strictly increasing sequence of natural numbers $(n_k)_{k=1}^{\infty}$, namely the place numbers of the terms that were kept from the original sequence to make the subsequence.

Summarizing: A subsequence of a sequence $(s_n)_{n=1}^{\infty}$ is a new sequence $(t_k)_{k=1}^{\infty}$, where $t_k = s_{n_k}$ for all k, and where $n_1 < n_2 < n_3 < \cdots$ are natural numbers. Or for short: a subsequence of a sequence $(s_n)_{n=1}^{\infty}$ is a new sequence $(s_{n_k})_{k=1}^{\infty}$, for natural numbers $n_1 < n_2 < n_3 < \cdots$.

Lemma 4.9. *If* $n_1 < n_2 < n_3 < \cdots$ *is a sequence of natural numbers then* $k \leq n_k$ *for all* $k \in \mathbb{N}$.

To say that a subsequence $(s_{n_k})_{k=1}^{\infty}$ converges to a number s, means that

$$\forall \epsilon > 0,\, \exists K > 0\, s.t.\ |s_{n_k} - s| < \epsilon \ \ \forall k \geq K.$$

Proposition 4.10. *If a sequence (s_n) converges to a number s, then every subsequence (s_{n_k}) also converges to s.*

Proof. Given $\epsilon > 0$ $\exists N$ s.t. $|s_n - s| < \epsilon$ $\forall n \geq N$. If $k \geq N$ then by Lemma 4.9, we have $n_k \geq k \geq N$, and so $|s_{n_k} - s| < \epsilon$. That is, $s_{n_k} \to s$. □

Theorem 4.11. (Bolzano-Weierstrass for sequences) *Every bounded sequence has a convergent subsequence.*

Theorem 4.12. *Every unbounded sequence contains a monotone subsequence with limit $+\infty$ or $-\infty$.*

Theorem 4.13. *A nonempty set S of real numbers is compact iff every sequence in S has a convergent subsequence with a limit in S.*

The limsup and liminf

Definition. The *limit superior* of a sequence (s_n), is the number

$$\text{limsup}_n\, s_n = \lim_{n\to\infty} \{\sup\{s_k : k \geq n\}\}.$$

The *limit inferior* is

$$\text{liminf}_n\, s_n = \lim_{n\to\infty} \{\inf\{s_k : k \geq n\}\}.$$

What the limsup and liminf are good for: First, they always exist, unlike the limit. For example, in the Example above, $\lim_n\ s_n$ does not exist. But we were able to compute the limsup and liminf. They always exist because as we saw in the example, they are limits of monotone sequences, which we know always exist. They behave similarly to the limit. That is, they obey laws analogous to the rules we saw in Section 4.1 above for limits. We will write down some of these laws momentarily. They can be used to check if the limit exists. In fact $\lim_n\ s_n$ exists iff $\text{liminf}_n\, s_n = \text{limsup}_n\, s_n$. So if $\text{liminf}_n\, s_n \neq \text{limsup}_n\, s_n$ then we may conclude that $\lim_n\ s_n$ does not exist. Finally, recall that in Calculus II there were certain tests ... One can improve these tests by using the limsup and liminf instead of the limit... .

Other properties of the limsup and liminf:

- In general, $\text{liminf}_n\, s_n \leq \text{limsup}_n\, s_n$. If $\lim_n\ s_n$ exists then $\text{liminf}_n\, s_n = \lim_n\ s_n = \text{limsup}_n\, s_n$.
- $\text{liminf}_n\, (-s_n) = -\text{limsup}_n\, s_n$.
- If $s_n \leq t_n$ for all n then $\text{limsup}_n\, s_n \leq \text{limsup}_n\, t_n$ and $\text{liminf}_n\, s_n \leq \text{liminf}_n\, t_n$.

- $\text{limsup}_n\,(Ks_n) = K\text{limsup}_n\, s_n$ and $\text{liminf}_n\,(Ks_n) = K\text{liminf}_n\, s_n$, if $K \geq 0$.
- ...

Summary: any sequence (s_n) has a limsup and a liminf (either a real number or $\pm\infty$), which behave like limits.

5. Limits and continuity of functions

5.1. **Limits of functions.** In this section $f : D \to \mathbb{R}$ is a function with domain $D \subset \mathbb{R}$, and let c be an accumulation point of D. We recall from Calculus I:

Definition: $\lim_{x\to c} f(x) = L$ if $\forall\epsilon > 0\ \exists\delta > 0$ s.t. $|f(x) - L| < \epsilon$ whenever $x \in D$ and $0 < |x - c| < \delta$.

In this case we say that L is the limit of f as x approaches c, and sometimes write $f(x) \to L$ as $x \to c$.

Definition: Now suppose that $c \in D$. We say that f is continuous at c if $\forall\epsilon > 0\ \exists\delta > 0$ s.t. $|f(x) - L| < \epsilon$ whenever $x \in D$ and $|x - c| < \delta$.

Theorem 5.1. (Main theorem # 1/MT # 1/Main theorem on limits) $\lim_{x\to c} f(x) = L$ *iff whenever* $x_n \to c$, *and* $x_n \neq c$ *for all* $n \in \mathbb{N}$, *then* $f(x_n) \to L$. *Here* (x_n) *is any sequence in the domain of* f.

Proof. ($\Rightarrow$) Suppose that $\lim_{x\to c} f(x) = L$, and that $x_n \to c, x_n \neq c$ for all $n \in \mathbb{N}$. So given $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x)-L| < \epsilon$ whenever $0 < |x-c| < \delta$. Since $x_n \to c$ there exists N such that $|x_n - c| < \delta$ if $n \geq N$. Also, $0 < |x_n - c|$ since $x_n \neq c$. Thus if $n \geq N$ then $|f(s_n) - L| < \epsilon$. That is, $f(s_n) \to L$.

($\Leftarrow$) We prove the contrapositive. Suppose that $\lim_{x\to c} f(x) \neq L$. So there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists x with $0 < |x - c| < \delta$ but $|f(x) - L| \geq \epsilon$. Let $\delta = \frac{1}{n}$ for $n \in \mathbb{N}$, so there exists x_n with $0 < |x_n - c| < \frac{1}{n}$ but $|f(x_n) - L| \geq \epsilon$. Since $|x_n - c| < \frac{1}{n} \to 0$ as $n \to \infty$, we have $x_n \to c$ by Fact 6 for sequences. Since $0 < |x_n - c|$, we have $x_n \neq c$. Since $|f(x_n) - L| \geq \epsilon$ for all n, the sequence $(f(x_n))$ does not converge to L. That is, we have found a sequence $x_n \to c, x_n \neq c$ for all $n \in \mathbb{N}$, but $(f(x_n))$ does not converge to L. □

Theorem 5.2. (Limit laws) *If* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = M$ *then*

- $\lim_{x\to c}(f(x) + g(x)) = L + M$.
- $\lim_{x\to c}(f(x) - g(x)) = L - M$.
- $\lim_{x\to c}(f(x)g(x)) = LM$.
- $\lim_{x\to c}(Kf(x)) = KL$, *if* K *is a constant.*
- $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ *if* $M \neq 0$.
- *If* $f(x) \leq g(x)$ *for all* x *in a deleted neighborhood of* c, *then* $L \leq M$.

- (Pinching/squeezing) *If* $f(x) \le g(x) \le r(x)$ *for all* x *in a deleted neighborhood of* c*, and if* $\lim_{x\to c} r(x) = L$*, then* $L = M$.

If $\lim_{y\to L} h(y) = K$*, and if there is a deleted neighborhood* N^* *of* c *such that* $f(N^*) \subset E \setminus \{L\}$*, where* E *is the domain of* h*, then* $\lim_{x\to c} h(f(x)) = K$.

Proof. All these proofs have the same proof technique, so we just do a few of them: They all begin with the statement: If $s_n \to c$, $s_n \neq c$ for all n, then $f(s_n) \to L$ (by Main theorem # 1 ($\Rightarrow$)). Similarly, $g(s_n) \to M$. By Fact 9(1) from Section 4.1 above, $f(s_n) + g(s_n) \to L + M$. By Main theorem # 1 ($\Leftarrow$) applied to $f + g$, $\lim_{x\to c} (f(x) + g(x)) = L + M$. This proves the first bullet.

Similarly, $\frac{f(s_n)}{g(s_n)} \to \frac{L}{M}$ by Fact 9(5) from Section 4.1 above. By Main theorem # 1 ($\Leftarrow$) applied to f/g, we $\lim_{x\to c} \frac{f(x)}{g(x)} = L/M$, the fifth bullet.

Similarly, if $f(x) \le g(x)$ as in the sixth bullet, then $f(s_n) \le g(s_n)$. Since $f(s_n) \to L$ and $g(s_n) \to M$, by Fact 4 in Section 4.1 above we have $L \le M$.

Similarly, if $f(x) \le g(x) \le r(x)$ as in the seventh bullet, then $f(s_n) \le g(s_n) \le r(s_n)$. Since $f(s_n) \to L$ and $r(s_n) \to L$ (by Main theorem # 1 ($\Rightarrow$) for f and for r), by Fact 5 in Section 4.1 above we have $g(s_n) \to L$. But we said above that $g(s_n) \to M$. So $L = M$.

Similarly, for the final assertion, if $t_n = f(s_n) \to L$ then $h(t_n) = h(f(s_n)) \to K$ by Main theorem # 1 ($\Rightarrow$) applied to h. So $\lim_{x\to c} h(f(x)) = K$ by Main theorem # 1 ($\Leftarrow$) applied to $h \circ f$. $\square$

• **Using Main theorem # 1 to show limits do not exist:** If you want to show that $\lim_{x\to c} f(x)$ does NOT exist, it is enough to either:

(a) Find a sequence $x_n \to c, x_n \neq c$ for all $n \in \mathbb{N}$, but $(f(x_n))$ does not converge; or

(b) Find two sequences $x_n \to c, y_n \to c, x_n \neq c$ for all $n \in \mathbb{N}$, and $y_n \neq c$ for all $n \in \mathbb{N}$, but $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$.

Proposition 5.3. *If* $\lim_{x\to c} f(x) = L > 0$*, then there is a deleted neighborhood* N^* *of* c *such that* $f(x) > L/2$ *for all* $x \in N^*$.

Proof. By definition of the limit, if $\epsilon = L/2 > 0$ then there exists a deleted neighborhood N^* of c such that $|f(x)-L| < \epsilon = L/2$ for all $x \in N^*$. But if $|f(x)-L| < L/2$ then $f(x) > L - L/2 = L/2 > 0$. $\square$

One-sided limits. If $f : (a, b) \to \mathbb{R}$ then we write $\lim\limits_{x\to a+} f(x) = L$ if $\forall\epsilon > 0\ \exists\delta > 0$ s.t. $|f(x) - L| < \epsilon$ whenever $a < x < a + \delta < b$. This is called the *right-hand limit* of f at a. Similarly for the left-hand limit $\lim\limits_{x\to b-} f(x) = L$, which means that $\forall\epsilon > 0\ \exists\delta > 0$ s.t. $|f(x) - L| < \epsilon$ whenever $a < b - \delta < x < b$.

The results above for usual (two-sided) limits have variants for one-sided limits, with essentially the same proofs. For example:

- Main theorem # 1 for right-hand limits: $\lim_{x\to a+} f(x) = L$ iff whenever $x_n \to a, x_n > a$ for all $n \in \mathbb{N}$, then $f(x_n) \to L$. Here (x_n) is a sequence in the domain of f.
- If $\lim_{x\to a+} f(x) = L$ and $\lim_{x\to a+} g(x) = M$ and $f(x) \leq g(x)$ on (a, b) then $L \leq M$.

Proposition 5.4. *If f is defined on a deleted neighborhood of c then $\lim\limits_{x\to c} f(x) = L$ iff $\lim\limits_{x\to c+} f(x) = \lim\limits_{x\to c-} f(x) = L$.*

5.2. **Continuous functions.** In this section $f : D \to \mathbb{R}$ is a function with domain $D \subset \mathbb{R}$, and let $c \in D$. We recall from Calculus I:

Definition: We say that f is continuous at c if $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $|f(x)-f(c)| < \epsilon$ whenever $x \in D$ and $|x - c| < \delta$.

We say that f is continuous on a set E inside its domain if f is continuous at every point in E. We simply say that f is continuous, if f is continuous at every point in its domain.

Theorem 5.5. (Main theorem # 2/MT # 2/Main theorem on continuity) *For f and c as above, the following are equivalent:*

(a) *f is continuous at c (that is, $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $|f(x) - f(c)| < \epsilon$ whenever $x \in D$ and $|x - c| < \delta$).*

(b) *Whenever (x_n) is a sequence in the domain of f with $x_n \to c$, then $f(x_n) \to f(c)$.*

(c) *For every neighborhood V of $f(c)$, $\exists$ a neighborhood U of c s.t. $f(U \cap D) \subseteq V$.*

If $c \in D'$ then these are also equivalent to:

(d) $\lim\limits_{x\to c} f(x) = f(c)$.

Proof. (a) $\Rightarrow$ (b) Suppose that f is continuous at c, and that $x_n \to c$. So given $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. Since $x_n \to c$ there exists N such that $|x_n - c| < \delta$ if $n \geq N$. Thus if $n \geq N$ then $|f(s_n) - f(c)| < \epsilon$. That is, $f(s_n) \to f(c)$.

(b) $\Rightarrow$ (a) We prove the contrapositive. Suppose that f is not continuous at c. So there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists x with $|x - c| < \delta$ but $|f(x) - f(c)| \geq \epsilon$. Let $\delta = \frac{1}{n}$ for $n \in \mathbb{N}$, so there exists x_n with $|x_n - c| < \frac{1}{n}$ but $|f(x_n) - f(c)| \geq \epsilon$. Since $|x_n - c| < \frac{1}{n} \to 0$ as $n \to \infty$, we have $x_n \to c$ by Fact 6 for sequences. Since $|f(x_n) - f(c)| \geq \epsilon$ for all n, the sequence $(f(x_n))$ does not converge to $f(c)$. That is, we have found a sequence $x_n \to c$, but $(f(x_n))$ does not converge to $f(c)$.

(a) $\Leftrightarrow$ (d) Simply write down the ϵ-δ definitions of (a) and (d) and it is clear that they are the same.

(a) $\Leftrightarrow$ (c) Saying "For every neighborhood V of $f(c)$" is the same as saying that "$\forall \epsilon > 0$, if $V = (f(c) - \epsilon, f(c) + \epsilon)$ then". Saying "$\exists$ a neighborhood U of c s.t." is the same as saying that "$\exists \delta > 0$ s.t. if $U = (c-\delta, c+\delta)$ then". Saying $f(U \cap D) \subseteq V$ is the same as saying that $f(x) \in V$ whenever $x \in U \cap D$, which is the same as saying that $|f(x) - f(c)| < \epsilon$ whenever $x \in D$ and $|x - c| < \delta$. So (c) is just saying that $\forall \epsilon > 0 \; \exists \delta > 0$ s.t. $|f(x) - f(c)| < \epsilon$ whenever $x \in D$ and $|x - c| < \delta$, which is (a). ☐

Theorem 5.6. (Continuity laws) *If f and g are both continuous at c then*

- *$f(x) \pm g(x)$ are continuous at c.*
- *$f(x)g(x)$ is continuous at c.*
- *$Kf(x)$ is continuous at c, if K is a constant.*
- *$\frac{f(x)}{g(x)}$ is continuous at c, if $g(c) \neq 0$.*

If h is a function which is continuous at $f(c)$, and if there is a neighborhood N of c such that $f(N) \subset E$, where E is the domain of h, then $h \circ f$ is continuous at c.

Proof. All these proofs have the same proof techniques as in Theorem 5.2, and so we just do a few of them: If $s_n \to c$, then $f(s_n) \to f(c)$ and $g(s_n) \to g(c)$ (by Main theorem # 2 (b)). By Fact 9(1) from Section 4.1 above, $f(s_n) + g(s_n) \to f(c) + g(c)$. By Main theorem # 2 ((b) $\Rightarrow$ (a)) applied to $f + g$, $f(x) + g(x)$ is continuous at c.

Similarly, $\frac{f(s_n)}{g(s_n)} \to \frac{f(c)}{g(c)}$ by Fact 9(5) from Section 4.1 above. By Main theorem # 2 ((b) $\Rightarrow$ (a)) applied to f/g, we have $\frac{f(x)}{g(x)}$, is continuous at c, the fourth bullet.

Similarly, for the final assertion, if $t_n = f(s_n) \to f(c)$ then $h(t_n) = h(f(s_n)) \to h(f(c))$ by Main theorem # 2 ((b)) applied to h. So $h(f(x))$ is continuous at c by Main theorem # 2 ((b) $\Rightarrow$ (a)) applied to $h \circ f$. ☐

One can use Main theorem # 2 (b) to show that functions are not continuous:

Test: $f : D \to \mathbb{R}$ is discontinuous at $c \in D$ iff there exists a sequence (s_n) in the domain of f s.t. $s_n \to c$ but $f(s_n) \not\to f(c)$.

Proposition 5.7. *If $f : (a, b) \to \mathbb{R}$ and $a < c < b$, and $f(c) > 0$, and f is continuous at c, then there is a neighborhood U of c such that $f(x) > f(c)/2$ for all $x \in U$.*

Finally, if f and g are continuous at c, then so are the functions $\max\{f(x), g(x)\}$ and $\min\{f(x), g(x)\}$.

5.3. **The Min-Max theorem and the IVT.** **Definition.** A function $f : D \to \mathbb{R}$ is called *bounded* if Range$(f) = f(D)$ is a bounded set. This is equivalent to: there are two constants m and M such that $m \leq f(x) \leq M$ for all $x \in D$. Equivalently, there is a constant K such that $|f(x)| \leq K$ for all $x \in D$. Similarly we say that f is bounded on a subset $E \subset D$ if $f(E)$ is a bounded set.

Theorem 5.8. (The Min-Max/Extreme Value Theorem) *If C is a compact set in $\mathbb{R}$, and if $f : C \to \mathbb{R}$ is continuous, then $f(C)$ is compact. In particular, f is a bounded function on C. Also, there exist $x_1, x_2 \in C$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in C$.*

Proof. We first use the criterion in Theorem 4.13 to show that $f(C)$ is compact. So let (y_n) be a sequence in $f(C)$. If $y_n = f(x_n)$ then (x_n) is a sequence in C, so by Theorem 4.13, there is a convergent subsequence $x_{n_k} \to x \in C$. By Main Theorem # 2 (b), $y_{n_k} = f(x_{n_k}) \to f(x)$, and $f(x) \in f(C)$. So by the criterion in Theorem 4.13 applied to $S = f(C)$, $f(C)$ is compact.

Any compact set in $\mathbb{R}$ is bounded. It also has a minimum and a maximum element, by Proposition 3.26, say m and M. Thus there exist $x_1, x_2 \in C$ such that $m = f(x_1) \leq f(x) \leq f(x_2) = M$ for all $x \in C$. □

The last theorem says that such functions have a *global* or *absolute* maximum on C (note that $f(x_2)$ is the absolute maximum value of f on C and $f(x_1)$ is the absolute minimum value of f on C). We said this in Calculus I.

Lemma 5.9. *Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $f(a) < 0 < f(b)$. Then $\exists c \in (a, b)$ s.t. $f(c) = 0$.*

Theorem 5.10. (The intermediate value theorem IVT) *If $f : [a, b] \to \mathbb{R}$ is continuous, and if z is a number strictly between $f(a)$ and $f(b)$ then there exists a $c \in (a, b)$ with $f(c) = z$.*

Corollary 5.11. (Existence of roots of positive numbers) *If $K \geq 0$ and $n \in \mathbb{N}$ then there exists $c \geq 0$ with $c^n = K$.*

Proof. We just do the square root, nth roots are similar. If $K = 0$ or $K = 1$ the result is obvious, so lets ignore these cases. Consider the function $f(x) = x^2$. There exists a positive number b such that $b^2 > K$ (indeed if $K > 1$ let $b = K$ and if $K < 1$ let $b = 1$). Thus $f(0) = 0 < K$ and $f(b) > K$, so by the IVT, there exists $c > 0$ with $f(c) = c^2 = K$. □

Corollary 5.12. $f : [a, b] \to \mathbb{R}$ *be continuous. Then $f([a, b])$ is a compact interval, hence there exists numbers c, d with $f([a, b]) = [c, d]$.*