Department of Mathematics, University of Houston
Math 4332. Intro to Real Analysis. David Blecher, Spring 2015
Homework 10.

As usual, exercises marked with * are to be turned in by the graduate students in the class.

(1) If f is a continuous scalar valued function on $\mathbb{R}$ which is of period 2π, prove that $\tilde{f}(e^{i\theta}) = f(\theta)$ for $\theta \in [0, 2\pi)$ defines a continuous function $\tilde{f}$ on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. The converse is much easier: show that if $\tilde{f}$ is a continuous function on this unit circle then $f(\theta) = \tilde{f}(e^{i\theta})$ is a continuous function on $\mathbb{R}$ which is of period 2π.

(2) Prove that $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx} dx = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$

(3) Prove that $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ may be written as $a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$. Conversely, show that $a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ can be written in the form $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$.

(4) (i) Let $f(x) = |x|$ for $-\pi \leq x < \pi$, and periodic of period 2π. Find the Fourier series of f.
(ii) Now let $f(x) = x^2$ for $-\pi \leq x \leq \pi$, and periodic period 2π. Find the Fourier series of f.
(iii) Now let $f(x) = 0$ on $[-\pi, 0)$, and $f(x) = 1$ on $[0, \pi)$, and periodic of period 2π. Find the Fourier series of f.

(5) Show that if functions $f_n \to f$ uniformly on $[a, b]$ then $f_n \to f$ in 2-norm on $[a, b]$.

(6*) Prove the complex scalar case of the Theorem on Best Approximation.

(7) If f is Riemann integrable on $[-\pi, \pi]$ prove (using the Corollary labelled as 'Parseval') that the Fourier series of f converges to f in 2-norm iff $\pi(2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2))$ (in complex case use $2\pi \sum_{k=-\infty}^{\infty} |c_k|^2$) equals $\|f\|_2^2$. (Complex case needs to be done by graduate students only.)

(8) Prove that if f is Riemann integrable on $[a, b]$, and if (φ_k) is an orthonormal sequence of functions on $[a, b]$ with $\sum_k b_k \varphi_k = f$ (convergence in 2-norm), then $b_n = \int_a^b f(x)\varphi_n(x)\, dx$ for each n (in the complex case we need a 'bar' over φ_n). Thus for example on $[a, b] = [-\pi, \pi]$ if $\sum_{k=-\infty}^{\infty} b_k e^{ikx} = f(x)$ (convergence in 2-norm) on $[-\pi, \pi]$ then $b_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\, dx$, so $\sum_{k=-\infty}^{\infty} b_k e^{ikx}$ is the Fourier series of f. Similarly, if $d_0 + \sum_{k=1}^{\infty} (d_k \cos(kx) + r_k \sin(kx)) = f$ (convergence in 2-norm) on $[-\pi, \pi]$ then $d_0 + \sum_{k=1}^{\infty} (d_k \cos(kx) + r_k \sin(kx))$ is just the Fourier series, and the d_k, r_k are just the Fourier coefficients (prove these too).