Department of Mathematics, University of Houston
Math 4332. Intro to Real Analysis. David Blecher, Spring 2015
Homework 10 Key.

As usual, exercises marked with * are to be turned in by the graduate students in the class.

(1) We do the converse first: if $\tilde{f}$ is a continuous function on the unit circle and $\gamma(\theta) = e^{i\theta} = (\cos\theta, \sin\theta)$ (see Homework 9 Question 1 and the matching class notes) then γ is continuous and of period 2π on $\mathbb{R}$, and so also $f = \tilde{f} \circ \gamma$, is continuous and of period 2π. [4 points for the above part.] For the other direction, suppose $\epsilon > 0$ is given and $w = e^{i\theta}$ and $\tilde{f}(w) = f(\theta) = \beta$. Choose a small $\delta > 0$ so that $|f(x) - f(\theta)| < \epsilon$ whenever $x \in (\theta - \delta, \theta + \delta)$. Then $V = \gamma((\theta - \delta, \theta + \delta))$ is clearly a small open arc containing w. For any $z \in V$ we have $z = \gamma(x)$ for some $x \in (\theta - \delta, \theta + \delta)$ so that $|\tilde{f}(z) - \tilde{f}(w)| = |f(x) - f(\theta)| < \epsilon$. So we have verified the ϵ definition of $\tilde{f}$ being continuous.

(2) If $k = 0$ this is obvious, otherwise $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx}dx = \frac{1}{ik}\frac{1}{2\pi}e^{ikx}]_{x=-\pi}^{\pi} = 0$. (Or one can do this like $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(kx)dx + i\int_{-\pi}^{\pi}\sin(kx)dx = \cdots$.) [3 points]

(3) First we suppose that a_k, b_k, c_k are any numbers. Then the first part follows because $c_k e^{ikx} = c_k\cos(kx) + ic_k\sin(kx)$, which equals $c_k\cos(|k|x) - ic_k\sin(|k|x)$ if $k < 0$. Thus $\sum_{k=-\infty}^{\infty} c_k e^{ikx} = c_0 + \sum_{k=1}^{\infty}((c_k + c_{-k})\cos(kx) + (ic_k - ic_{-k})\sin(kx))$. Similarly, the second part follows because $a_n\cos nx = \frac{1}{2}a_n e^{inx} + \frac{1}{2}a_n e^{-inx}$, and $b_n\sin nx = \frac{1}{2i}b_n e^{inx} - \frac{1}{2i}b_n e^{-inx}$. Thus $a_0 + \sum_{k=1}^{\infty}(a_n\cos nx + b_n\sin nx) = a_0 + \frac{1}{2}\sum_{k=1}^{\infty}((a_n - ib_n)e^{inx} + (a_n + ib_n)e^{-inx})$. [2 points for completeness only]

If now a_k, b_k, c_k are the usual Fourier coefficients of x, then we have that

$$\frac{1}{2}(a_k - ib_k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx - i\int_{-\pi}^{\pi} f(x)\sin(kx)dx) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos(kx) - i\sin(kx))dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx = c_k.$$

That is, $\frac{1}{2}(a_k - ib_k) = c_k$. Similarly, $a_k + ib_k = c_{-k}$. Clearly $a_0 = c_0$. So the line at the end of the last paragraph actually says that $a_0 + \sum_{k=1}^{\infty}(a_n\cos nx + b_n\sin nx) = c_0 + \sum_{k=1}^{\infty} c_n e^{inx} + c_{-n}e^{-inx} = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$.

(4) (i) This is an even function, and so $f(x)\sin(kx)$ is odd, so $b_k = 0$ for all $k \in \mathbb{N}$. Similarly, $f(x)\cos(kx)$ is even, so if $k \in \mathbb{N}$ then $a_k = \frac{2}{\pi}\int_0^{\pi} x\cos(kx)dx = \frac{2}{\pi}\frac{(-1)^n - 1}{n^2}$ by Calculus (integration by parts). Clearly $a_0 = \frac{1}{\pi}\int_0^{\pi} xdx = \frac{\pi}{2}$. So the Fourier series of f is $\frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos((2k-1)x)}{(2k-1)^2}$. [5 points]
(ii) Again this is an even function, and so for all $k \in \mathbb{N}$ we have $b_k = 0$ as in (i), and $a_k = \frac{2}{\pi}\int_0^{\pi} x^2\cos(kx)dx$, which again one can do by Calculus (integration by parts) to get $a_k = 4\frac{(-1)^k}{k^2}$. Clearly $a_0 = \frac{1}{\pi}\int_0^{\pi} x^2dx = \frac{\pi^2}{3}$. So the Fourier series is $\frac{\pi^2}{3} + 4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos(kx)$. [4 points]
(iii) For all $k \in \mathbb{N}$ we have

$$b_k = \frac{1}{\pi}\int_0^{\pi}\sin(kx)dx = -\frac{1}{k\pi}\cos(kx)]_0^{\pi} = \frac{1 - (-1)^k}{k\pi},$$

and $a_k = \frac{1}{\pi}\int_0^{\pi}\cos(kx)dx = 0$. Clearly $a_0 = \frac{1}{2\pi}\int_0^{\pi} 1dx = \frac{1}{2}$. So the Fourier series of f is $\frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin((2k-1)x)}{2k-1}$. [4 points]

(5) This follows because $\int_a^b |f_n - f|^2dx \leq \int_a^b \|f_n - f\|_\infty^2 dx = (b-a)\|f_n - f\|_\infty^2$. That is, $\|f_n - f\|_2 \leq \sqrt{b-a}\|f_n - f\|_\infty$. If $f_n \to f$ uniformly on $[a, b]$ then $\|f_n - f\|_2 \leq \sqrt{b-a}\,\|f_n - f\|_\infty \to 0$ as $n \to \infty$, so by 'squeezing' $\|f_n - f\|_2 \to 0$. That is, $f_n \to f$ in 2-norm on $[a, b]$. [4 points]

(6*) The complex scalar case of the Theorem on Best Approximation was indicated in class by green ink marking the terms that need to be 'conjugated' (the conjugate of z being $\bar{z}$), up to a certain part of the proof. The next few steps in the complex scalar case should read

$$\|f\|_2^2 + \sum_{k=1}^{N}|b_k|^2 - \int_a^b f\overline{\sum_{k=1}^{N} b_k\phi_k}dx - \int_a^b (\sum_{k=1}^{N} b_k\phi_k)\bar{f}dx = \|f\|_2^2 + \sum_{k=1}^{N}|b_k|^2 - w - \bar{w},$$

where

$$w = \int_a^b f\overline{\sum_{k=1}^{N} b_k\phi_k}dx = \sum_{k=1}^{N}\overline{b_k}\int_a^b f\,\overline{\phi_k}dx = \sum_{k=1}^{N}\overline{b_k}c_k.$$

Thus we are looking at

$$\|f\|_2^2 + \sum_{k=1}^{N}|b_k|^2 - \sum_{k=1}^{N}\overline{b_k}c_k - \sum_{k=1}^{N} b_k\overline{c_k} = \|f\|_2^2 + \sum_{k=1}^{N}(|b_k|^2 - \overline{b_k}c_k - b_k\overline{c_k} + |c_k|^2 - |c_k|^2).$$

The rest is as in class.

(7) This is basically just the Corollary labelled 'Parseval' at the end of class on March 24, in the particular case of the usual orthonormal set on $[-\pi,\pi]$. In the real case this orthonormal set on $[-\pi,\pi]$ is $\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos nx, \frac{1}{\sqrt{\pi}}\sin nx : n=1,2,\cdots\}$, and so the (c_n) sequence in that Corollary labelled 'Parseval' is:

$$c_1 = \int_{-\pi}^{\pi} f(x)\,(\frac{1}{\sqrt{2\pi}})dx = \sqrt{2\pi}\,a_0;\; c_2 = \int_{-\pi}^{\pi} f(x)\,(\frac{1}{\sqrt{\pi}}\cos x)dx = \sqrt{\pi}\,a_1;\; c_3 = \int_{-\pi}^{\pi} f(x)\,(\frac{1}{\sqrt{\pi}}\sin x)dx = \sqrt{\pi}\,b_1,$$

and so on, for example $c_4 = \int_{-\pi}^{\pi} f(x)\,(\frac{1}{\sqrt{\pi}}\cos x)dx = \sqrt{\pi}\,a_2, c_5 = \sqrt{\pi}\,b_2, c_6 = \sqrt{\pi}\,a_3$, etc. Then

$$\sum_{k=1}^{\infty} |c_k|^2 = 2\pi a_0^2 + \pi a_1^2 + \pi b_1^2 + \pi a_2^2 + \pi b_2^2 + \cdots = \pi(2a_0^2 + a_1^2 + b_1^2 + a_2^2 + b_2^2 + \cdots) = \pi(2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)).$$

Also,

$$\sum_{k=1}^{2N+1} c_k\,\phi_k = \sqrt{2\pi}\,a_0\frac{1}{\sqrt{2\pi}} + \sqrt{\pi}\,a_1\frac{1}{\sqrt{\pi}}\cos x + \sqrt{\pi}\,b_1\frac{1}{\sqrt{\pi}}\sin x + \cdots + \sqrt{\pi}\,a_n\frac{1}{\sqrt{\pi}}\cos(Nx) + \sqrt{\pi}\,b_n\frac{1}{\sqrt{\pi}}\sin(Nx),$$

which is just the Nth partial sum of the Fourier series of f. So by the Corollary labelled 'Parseval' in class, $\|f\|_2^2 = \pi(2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2))$ iff the Nth partial sum of the Fourier series of f converges to f in 2-norm. [3 points for completeness only]

In the complex case (for graduate students only) there is a notational issue, since the c_n in that Corollary labelled 'Parseval' is different from the usual nth complex Fourier coefficient usually also called c_n. To avoid the ambiguity, lets write the c_n in that Corollary labelled 'Parseval' upper case: C_n. Here the appropriate orthonormal set on $[-\pi,\pi]$ is $\{\frac{1}{\sqrt{2\pi}}e^{inx} : n = \cdots,-2,-1,0,1,2,\cdots\}$, and it is easy to see that

$$C_1 = \sqrt{2\pi}\,c_0, C_1 = \sqrt{2\pi}\,c_1, C_2 = \sqrt{2\pi}\,c_{-1}, C_3 = \sqrt{2\pi}\,c_2, C_4 = \sqrt{2\pi}\,c_{-2}, C_5 = \sqrt{2\pi}\,c_3, \cdots.$$

Then

$$\sum_{k=1}^{\infty} |C_k|^2 = 2\pi \sum_{k=-\infty}^{\infty} |c_k|^2,$$

Also,

$$\sum_{k=1}^{2N+1} C_k\,\phi_k = \sum_{k=-N}^{N} \sqrt{2\pi}\,c_k(\frac{1}{\sqrt{2\pi}}e^{ikx}) = \sum_{k=-N}^{N} c_k e^{ikx},$$

which is just the Nth partial sum of the Fourier series of f. So by the Corollary labelled 'Parseval' in class, $\|f\|_2^2 = 2\pi\sum_{k=-\infty}^{\infty}|c_k|^2$ iff the Nth partial sum of the Fourier series of f converges to f in 2-norm.

(8) Suppose $\sum_{k=1}^{\infty} b_k\varphi_k = f$ (convergence in 2-norm), Then

$$\int_a^b (\sum_{k=1}^{N} b_k\varphi_k)\,\overline{\varphi_j}\,dx = \sum_{k=1}^{N} b_k \int_a^b \varphi_k\,\overline{\varphi_j}\,dx = b_j$$

if $N \geq j$. Hence $\int_a^b f\,\overline{\varphi_j}\,dx - b_j = \int_a^b (f - \sum_{k=1}^{N} b_k\varphi_k)\,\overline{\varphi_j}\,dx$ if $N \geq j$, so by Cauchy-Schwarz,

$$|\int_a^b f\,\overline{\varphi_j}\,dx - b_j| = |\int_a^b (f - \sum_{k=1}^{N} b_k\varphi_k)\,\overline{\varphi_j}\,dx| \leq \|f - \sum_{k=1}^{N} b_k\varphi_k\|_2\,\|\overline{\varphi_j}\|_2 \to 0$$

as $N \to \infty$. Thus $b_j = \int_a^b f\,\overline{\varphi_j}\,dx$. [8 points]

The two assertions about the case that $[a,b] = [-\pi,\pi]$ can either be done by very slight variants of the argument in the last paragraph, or can be deduced from the last paragraph as in Question 7.