Department of Mathematics, University of Houston
Math 4332. Intro to Real Analysis. David Blecher, Spring 2015
Homework 11.

As usual, exercises marked with * are to be turned in by the graduate students in the class.

(0) Prove that if we have a sequence of Riemann integrable functions $h, h_1, h_2, h_3, \cdots$ which are Riemann integrable on $[a, b]$, and if $h_n \to h$ in 2-norm on $[a, b]$, and $a \le c < d \le b$, then $\int_c^d h_n \, dx \to \int_c^d h \, dx$. [Hint: use the Cauchy-Schwarz inequality for integrals.]

(1) Suppose f is a real valued differentiable 2π-periodic function with f' Riemann integrable on $[-\pi, \pi]$. Prove that the sum of the Fourier coefficients of f is absolutely convergent, and the Fourier series of f converges uniformly to f on $[-\pi, \pi]$. [Hint: deduce this from (or by a modification of the proof of) the 'complex case' of the same result done in class.]

(2) Prove that the convolution on $[-\pi, \pi]$ satisfies $f * g = g * f$ (here as in the notes, f, g are 2π-periodic functions which are Riemann integrable on $[-\pi, \pi]$).

(3) Show that (a) $x = \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ if $0 < x < 2\pi$.
(b) Deduce from (a) that $\frac{x^2}{2} = \pi x - \frac{\pi^2}{3} + 2\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ if $0 \le x \le 2\pi$. You may use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.
(c) $\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{2n-1}$ if $0 < x < \pi$.
(d) $\cos x = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{n\sin(2nx)}{4n^2-1}$ if $0 < x < \pi$.
(e) Deduce from (c) that $\frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2n-1)^2}$, and that $\frac{\pi}{4}(\frac{\pi}{2} - x) = \sum_{k=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2}$ for $0 \le x \le \pi$.

(4) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and 2π-periodic. Suppose also that $\sum_{k=1}^{\infty} k(|a_k| + |b_k|) < \infty$ (in the complex case $\sum_{k=-\infty}^{\infty} |k\, c_k| < \infty$), where a_k, b_k, c_k are the Fourier coefficients of f. Prove that f is differentiable, and that the Fourier series for f converges uniformly to f.

(5*) On Cesaro sums (see e.g. Homework 5 Question 7 for notation): Graduate students only: look up a proof of the following Fact: if a sequence (s_n) is Cesáro summable and its Cesáro sum is s, and if $(n(s_n - s_{n-1}))$ is a bounded sequence then $s_n \to s$ as $n \to \infty$. Print out and attach the proof, or write it in your own words.

(6) Show that if a function is differentiable at a point x then it satisfies the Lipschitz condition at x mentioned above Theorem 4.4 in the classnotes. (Hint: look at the ϵ-δ definition of $\lim_{y\to x} \frac{f(y)-f(x)}{y-x} = f'(x)$, with $\epsilon = 1$.)