Department of Mathematics, University of Houston
Math 4332. Intro to Real Analysis. David Blecher, Spring 2015
Homework 11 Key.

As usual, exercises marked with * are to be turned in by the graduate students in the class.

(0) This is now a Proposition towards the end of the online classnotes. It could be asked on Test 2.

(1) By the complex case done in class, $\sum_{k=-\infty}^{\infty} |c_k| < \infty$. If $n \in \mathbb{N}$ then $a_n = c_n + c_{-n}$, and $b_n = i(c_n - c_{-n})$, so that $\sum_{k=1}^{\infty} (|a_n| + |b_n|) \leq \sum_{k=1}^{\infty} 2(|c_n| + |c_{-n}|) < \infty$. The rest follows as in the complex case, i.e. from Corollary 4 in that section. [6]

(2) The convolution on $[-\pi, \pi]$ is $(f * g)(x) = \int_{-\pi}^{\pi} f(y)g(x-y)dy$. Let $u = x - y$ for fixed x, then $du = -dy$, and $y = x - u$, and the integral becomes

$$-\int_{x+\pi}^{x-\pi} f(x-u)g(u)dy = \int_{x-\pi}^{x+\pi} g(u)f(x-u)dy = \int_{-\pi}^{\pi} g(u)f(x-u)dy = (g * f)(x).$$

In the last integral we have used the fact that $g(u)f(x-u)$ is 2π-periodic, and for any c-periodic function h and any real numbers d and b, we have $\int_d^{d+c} h dt = \int_b^{b+c} h dt$. [7]

(3) (a) Make the function x on $[0, 2\pi)$ to be periodic of period 2π by just repeating it endlessly. Call this 2π-periodic function f. So e.g. $f(x) = x + 2\pi$ for $-\pi < x < 0$. The Fourier series on $[-\pi, \pi]$ is easy to compute: it is $\pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$. By one of the Theorems or Corollaries in the Pointwise Convergence Section of the notes, this series converges pointwise to $f(x)$ on $(0, \pi]$ and on $[-\pi, 0)$, hence by periodicity on $[\pi, 2\pi)$. Thus $x = \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ if $0 < x < 2\pi$.

(b) If $0 \leq x \leq 2\pi$ then integrate $\int_0^x$ in (a). We have to use Question 0 to integrate the series in (a). We get

$$\frac{x^2}{2} = \pi x - \int_0^x (2\sum_{n=1}^{\infty} \frac{\sin nx}{n})dx = \pi x - 2\sum_{n=1}^{\infty} \int_0^x \frac{\sin nx}{n} dx = \pi x - 2\sum_{n=1}^{\infty} (\frac{\cos nx}{n^2} - \frac{1}{n^2}).$$

But $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (this may be seen for example by applying the Parseval formula to the function in (a)). So $\frac{x^2}{2} = \pi x - \frac{\pi^2}{3} + 2\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ if $0 \leq x \leq 2\pi$.

(c) Multiplying by $\frac{\pi}{4}$ in Homework 10 Question 4 (iii) we get the Fourier series $\frac{\pi}{8} + \frac{1}{2}\sum_{k=1}^{\infty} \frac{\sin((2k-1)x)}{2k-1}$. By one of the Theorems or Corollaries in the Pointwise Convergence Section of the notes, this series converges pointwise to $\frac{\pi}{4}$ if $0 < x < \pi$. Then subtract $\frac{\pi}{8}$ and multiply by 2. [8]

(d) Let $f(x) = \cos x$ if $0 < x < \pi$, and $f(x) = -\cos x$ if $-\pi < x < 0$. This is an odd function, and its Fourier series on $[-\pi, \pi]$ is easy to compute: it is $\frac{8}{\pi}\sum_{n=1}^{\infty} \frac{n \sin(2nx)}{4n^2-1}$. By one of the Theorems or Corollaries in the Pointwise Convergence Section of the notes, this series converges pointwise to $\cos x$ if $0 < x < \pi$. [3] for completeness only

(e) $0 \leq x \leq \pi$ then integrate between x and $\frac{\pi}{2}$ in (c). If $0 \leq x \leq \frac{\pi}{2}$ we get, using Question 0 to integrate the series similarly to Question (b) above:

$$\frac{\pi}{4}(\frac{\pi}{2} - x) = \int_x^{\frac{\pi}{2}} (\sum_{n=1}^{\infty} \frac{\sin((2n-1)t)}{2n-1})dt = \sum_{n=1}^{\infty} \int_x^{\frac{\pi}{2}} \frac{\sin((2n-1)t)}{2n-1} dt = -\sum_{n=1}^{\infty} \frac{\cos((2n-1)t)}{(2n-1)^2}]_x^{\frac{\pi}{2}} = \sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2}.$$

If $\frac{\pi}{2} \leq x \leq \pi$ we get similarly,

$$\frac{\pi}{4}(x - \frac{\pi}{2}) = \sum_{n=1}^{\infty} \int_{\frac{\pi}{2}}^x \frac{\sin((2n-1)t)}{2n-1} dt = -\sum_{n=1}^{\infty} \frac{\cos((2n-1)t)}{(2n-1)^2}]_{\frac{\pi}{2}}^x = -\sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2}.$$

Thus

$$\frac{\pi}{4}(\frac{\pi}{2} - x) = \sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2}, \qquad 0 \leq x \leq \pi.$$

Setting $x = 0$ gives $\frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2n-1)^2}$. [9]

(4) We just do the complex case, the real case is similar. Since $|ikc_k e^{ikx}| \leq |kc_k|$, if $\sum_{k=-\infty}^{\infty} |kc_k| < \infty$ then by the Weierstrass M-test the series $\sum_{k=-\infty}^{\infty} ikc_k\, e^{ikx}$ converges uniformly to a function g. (Strictly speaking, this result was phrased for series $\sum_{k=1}^{\infty}$ rather than $\sum_{k=-\infty}^{\infty}$, but the latter can easily be rewritten as the former). Similarly, the sum of the Fourier coefficients of f converges absolutely by the Comparison Test, since $|c_k| \leq |kc_k|$. By Corollary 4 in the notes, the Fourier series for f converges uniformly to f. By the theorem on derivatives of series of functions, $f'(x) = \sum_{k=-\infty}^{\infty} ikc_k\, e^{ikx} = g(x)$. [12]

(6) Indeed the ϵ-δ definition of $\lim_{y\to x} \frac{f(y)-f(x)}{y-x} = f'(x)$, with $\epsilon = 1$, says that there is a $\delta > 0$ such that

$$|\frac{f(y)-f(x)}{y-x} - f'(x)| = |\frac{f(y)-f(x)-f'(x)(y-x)}{y-x}| < 1, \qquad |y-x| < \delta.$$

Multiplying by $|y-x|$ and using the triangle inequality shows that if $|y-x| < \delta$,

$$|f(y)-f(x)| \le |f(y)-f(x)-f'(x)(y-x)| + |f'(x)(y-x)| < |y-x| + |f'(x)(y-x)| = M|y-x|,$$

where $M = 1 + |f'(x)|$.)