Department of Mathematics, University of Houston
Math 4332. Intro to Real Analysis. David Blecher, Spring 2015
Homework 12.

As usual, exercises marked with * are to be turned in by the graduate students in the class.

(1) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(x, y) = (\sin x \cos y, \sin x \sin y, \cos x \cos y)$. Find $f'(x, y)$.

(2) If A is an $m \times n$ matrix, $\vec{b} \in \mathbb{R}^m$, and $\vec{a} \in \mathbb{R}^n$, then find $f'(\vec{x})$ if

(i) $f(\vec{x}) = A\vec{x} + \vec{b}$,
(ii) $f(\vec{x}) = (\vec{a}^T \vec{x})\, \vec{b}$,
(iii) $f(\vec{x}) = \vec{x}$.

(3) Find $f'(0, 0)$ if it exists if

(i) $f(x, y) = x^2y^2 \log(x^2 + y^2)$ if $(x, y) \neq (0, 0)$, $f(0, 0) = 0$.
(ii) $f(x, y) = xy \sin(\frac{1}{x^2+y^2})$ if $(x, y) \neq (0, 0)$, $f(0, 0) = 0$.
(iii) $f(x, y) = \frac{x^2+y^2}{\sin(\sqrt{x^2+y^2})}$ if $(x, y) \neq (0, 0)$, $f(0, 0) = 0$.
(iv) $f(x, y) = \sqrt{|xy|}$.
(v) $f(\vec{x}) = \|\vec{x}\|_2^\alpha$ for $\alpha \geq 1$.

(4) Compute $\frac{\partial f}{\partial x}$ and determine where it is continuous, where $f(x, y) = \frac{x^4+y^4}{x^2+y^2}$ if $(x, y) \neq (0, 0)$, $f(0, 0) = 0$.

(5) Suppose $f : S \to \mathbb{R}$ is differentiable on an open set $S \subseteq \mathbb{R}^n$, and suppose f has a local maximum at $\vec{x} \in S$ (so $\exists \delta > 0$ at $f(\vec{x}) \geq f(\vec{y})$ for all $\vec{y} \in B(\vec{x}, \delta)$). Prove $f'(\vec{x}) = \vec{0}$.

(6) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $g(x, y, z) = x^2 + y^2 + z^2$. Show that if $h = f \circ g$ then $\|h'(x, y, z)\|^2 = 4g(x, y, z)\, f'(g(x, y, z))^2$.

(7) Write out the Calculus III chain rule for $\frac{\partial h}{\partial x}$, using our chain rule if $h(x) = f(u(x, y, z), v(x, y), w(x))$.

(8) Let $f(u, v, w) = (e^{u-w}, \cos(v + u) + \sin(u + v + w))$ and $g(x, y) = (e^x, \cos(y - x), e^{-y})$. Calculate $(f \circ g)'(0, 0)$ using our chain rule.

(9) Let $f : \mathbb{R} \to \mathbb{R}^2$ be given by $f(t) = (\cos t, \sin t)$. Show that f' is continuous. Prove that there is no $z \in \mathbb{R}$ with $f(2\pi) - f(0) = f'(z)(2\pi - 0)$.