Department of Mathematics, University of Houston
Math 4332. Intro to Real Analysis. David Blecher, Spring 2015
Homework 12 Solutions

(1) Since f is C^1, by a theorem from class this will just be the Jacobian matrix for f, namely $\begin{bmatrix} \cos x \cos y & -\sin x \sin y \\ \cos x \sin y & \sin x \cos y \\ -\sin x \cos y & -\cos x \sin y \end{bmatrix}$.

(2) (i) $f' = A$, since $\lim_{\vec{h} \to \vec{0}} \frac{f(\vec{x}+\vec{h})-f(\vec{x})-A\vec{h}}{\|\vec{h}\|_2} = \lim_{\vec{h} \to \vec{0}} \frac{A(\vec{x}+\vec{h})-A\vec{x}-A\vec{h}}{\|\vec{h}\|_2} = \lim_{\vec{h} \to \vec{0}} \frac{\vec{0}}{\|\vec{h}\|_2} = \vec{0}$.

(ii) $f(\vec{x}) = \vec{b}\,(\vec{a}^T\vec{x}) = A\vec{x}$ where $A = \vec{b}\,\vec{a}^T$, so by (i) we have $f' = A = \vec{b}\,\vec{a}^T$.

(iii) $f(\vec{x}) = A\vec{x}$ where $A = I$, so by (i) we have $f' = A = I$.

(3) (i) One can easily check (as e.g. in Q 4 below) $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$, so either $f'(0,0) = [0\ 0]$ or $f'(0,0)$ does not exist, depending on whether $\lim_{\vec{h} \to \vec{0}} \frac{f(\vec{h})-f(\vec{0})-[0\ 0]\vec{h}}{\|\vec{h}\|_2}$ is zero or not. But this limit is $\lim_{\vec{h} \to \vec{0}} \frac{1}{\|\vec{h}\|_2} |h_1^2 h_2^2 \log \|\vec{h}\|_2|$, which is 0, since

$$\frac{1}{\|\vec{h}\|_2} |h_1^2 h_2^2 \log \|\vec{h}\|_2| \leq \frac{1}{\|\vec{h}\|_2} \|\vec{h}\|_2^4 \,|\log \|\vec{h}\|_2| = 2\|\vec{h}\|_2^3 \,|\log \|\vec{h}\|| \to 0$$

as $\vec{h} \to \vec{0}$.

(ii) Similarly as (i), $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$, so either $f'(0,0) = [0\ 0]$ or $f'(0,0)$ does not exist, depending on whether $\lim_{\vec{h} \to \vec{0}} \frac{f(\vec{h})-f(\vec{0})-[0\ 0]\vec{h}}{\|\vec{h}\|_2}$ is zero or not. But this limit is $\lim_{\vec{h} \to \vec{0}} \frac{1}{\|\vec{h}\|_2} h_1 h_2 \sin(\frac{1}{\|\vec{h}\|_2^2})$, which is 0, since

$$\frac{1}{\|\vec{h}\|_2} |h_1 h_2 \sin(\frac{1}{\|\vec{h}\|_2^2})| \leq \frac{1}{\|\vec{h}\|_2} \|\vec{h}\|_2^2 = \|\vec{h}\|_2 \to 0$$

as $\vec{h} \to \vec{0}$.

(iii) Here $\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{h}{\sin|h|}$ which does not exist. So $f'(0,0)$ does not exist.

(iv) One can easily check (as e.g. in Q 4 below) $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$, so either $f'(0,0) = [0\ 0]$ or $f'(0,0)$ does not exist, depending on whether $\lim_{\vec{h} \to \vec{0}} \frac{f(\vec{h})-f(\vec{0})-[0\ 0]\vec{h}}{\|\vec{h}\|_2}$ is zero or not. But this limit is $\lim_{\vec{h} \to \vec{0}} \frac{\sqrt{|h_1 h_2|}}{\sqrt{h_1^2+h_2^2}}$ which is not zero (for example if $h_1 = h_2 = t$ as $t \searrow 0$, then the limit is $\lim_{t \to 0} \frac{\sqrt{t^2}}{\sqrt{2t^2}} = \lim_{t \to 0} \frac{1}{\sqrt{2}} \neq 0$.)

(v) There are two cases $\alpha > 1$ and $\alpha = 1$. If $\alpha > 1$ then one can easily check (as e.g. in Q 4 below) that $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$, so either $f'(0,0) = [0\ 0]$ or $f'(0,0)$ does not exist, depending on whether $\lim_{\vec{h} \to \vec{0}} \frac{f(\vec{h})-f(\vec{0})-[0\ 0]\vec{h}}{\|\vec{h}\|_2}$ is zero or not. But this limit is $\lim_{\vec{h} \to \vec{0}} \frac{\|\vec{h}\|_2^\alpha}{\|\vec{h}\|_2} = \lim_{\vec{h} \to \vec{0}} \|\vec{h}\|_2^{\alpha-1} = 0$. So $f'(0,0) = [0\ 0]$. If $\alpha = 1$ then one can easily check $\frac{\partial f}{\partial x}(0,0)$ does not exist. So $f'(0,0)$ does not exist.

(4) Here $\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{h^2-0}{h} = 0$. We know the quotient of two continuous functions is continuous at any point at which the denominator is nonzero. So f is continuous except possibly at $(0,0)$. But it is also continuous at $(0,0)$ since $|f(x,y) - 0| \leq 2\frac{\|(x,y)\|_2^4}{\|(x,y)\|_2^2} \leq 2\|(x,y)\|_2^2 \to 0$ as $(x,y) \to \vec{0}$.

(5) It is enough to prove that $\frac{\partial f}{\partial x_k}(\vec{x}) = 0$ for all k. We do this for $k = 1$, the others are similar. The argument is as in 3333: Note $f(\vec{x} + h\vec{e}_1) \leq f(\vec{x})$ so that if $h > 0$ then $\frac{f(\vec{x}+h\vec{e}_1)-f(\vec{x})}{h} \leq 0$, and so $\lim_{h \to 0^+} \frac{f(\vec{x}+h\vec{e}_1)-f(\vec{x})}{h} \leq 0$. If $h < 0$ then $\frac{f(\vec{x}+h\vec{e}_1)-f(\vec{x})}{h} \geq 0$, and so $\lim_{h \to 0^-} \frac{f(\vec{x}+h\vec{e}_1)-f(\vec{x})}{h} \geq 0$. So $\frac{\partial f}{\partial x_1}(\vec{x}) = \lim_{h \to 0} \frac{f(\vec{x}+h\vec{e}_1)-f(\vec{x})}{h} = 0$.

(6) By the chain rule, $h'(x,y,z) = f'(g(x,y,z))g'(x,y,z)$. Now $g'(x,y,z) = [2x\ 2y\ 2z]$. Hence

$$\|h'(x,y,z)\|_2^2 = (f'(g(x,y,z)))^2 \|[2x\ 2y\ 2z]\|_2 = 4f'(g(x,y,z))^2(x^2+y^2+z^2) = 4g(x,y,z)\ f'(g(x,y,z))^2.$$

(7) The chain rule is $\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{dw}{dx}$. For this we need h, f, u, v, w to be differentiable, The justification goes as follows: let $g(x,y,z) = (u(x,y,z), v(x,y), w(x))$. Then the derivative of g is

$$g'(x,y,z) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & 0 \\ \frac{dw}{dx} & 0 & 0 \end{bmatrix}.$$

By the chain rule we have $h' = f'(g(x,y,z))\, g'(x,y,z)$. That is,

$$[\frac{\partial h}{\partial x}\ \frac{\partial h}{\partial y}\ \frac{\partial h}{\partial z}] = [\frac{\partial f}{\partial u}\ \frac{\partial f}{\partial v}\ \frac{\partial f}{\partial w}] \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & 0 \\ \frac{dw}{dx} & 0 & 0 \end{bmatrix}.$$

Multiplying these matrices, and looking at the 1-1 entry, we see $\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{dw}{dx}$ as desired.

(8) We have

$$f' = \begin{bmatrix} e^{u-w} & 0 & -e^{u-w} \\ -\sin(v+u)+\cos(u+v+w) & -\sin(v+u)+\cos(u+v+w) & \cos(u+v+w) \end{bmatrix}$$

and

$$g' = \begin{bmatrix} e^{x} & 0 \\ \sin(y-x) & -\sin(y-x) \\ 0 & -e^{-y} \end{bmatrix}.$$

Now $g(0,0) = (1,1,1)$ and

$$g'(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix},$$

$$f'(g(0,0)) = f'(1,1,1) = \begin{bmatrix} 1 & 0 & -1 \\ -\sin(2)+\cos(3) & -\sin(2)+\cos(3) & \cos(3) \end{bmatrix}.$$

By the chain rule

$$D(f\circ g)(0,0) = f'(g(0,0))\, g'(0,0) = \begin{bmatrix} 1 & 1 \\ -\sin(2)+\cos(3) & -\cos(3) \end{bmatrix}.$$

(9) Note $f' = (-\sin t, \cos t)$, which is continuous by 3334/4332. It is also never $\vec{0}$, so the right side of the expession is never zero. But the left side is 0 since f is 2π-periodic.