Department of Mathematics, University of Houston
Math 4332. Intro to Real Analysis. David Blecher, Spring 2015
Homework 13 Key.

As usual, exercises marked with * are to be turned in by the graduate students in the class.

(1) $f(0,1) = (-1,0)$, and $f' = \begin{bmatrix} 3u^2 & -2v \\ \cos u & -1/v \end{bmatrix}$, which equals $\begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix}$ at $(0,1)$. The determinant of this is $2 \neq 0$, so f^{-1} exists in a neighborhood of $(-1,0)$, and $(f^{-1})'(-1,0) = \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix}^{-1}$, which is easy to compute.

(2) (i) The Jacobian matrix at $(0,0,0)$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, which has nonzero determinant ($= 1$). Thus by the inverse function theorem, it is possible to solve for x, y, and z explicitly in terms of u, v, w, in a neighborhood of the point $(x, y, z) = (0, 0, 0)$.

(ii) The Jacobian matrix of derivatives of x, y, and z with respect to u, v, w, at $(0,0,0)$ is the inverse matrix of the matrix in (i), that is, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

(iii) Reading the first row of the last matrix, we have $\frac{\partial x}{\partial u} = 1$, $\frac{\partial x}{\partial v} = \frac{\partial x}{\partial w} = 0$.

(3) $f' = [6x^2 - 6x \quad 6y^2 + 6y]$. Since we wish to solve for y in terms of x by the implicit function, we look at $A = 6y^2 + 6y$, which is not invertible exactly iff $y = 0$ or $y = -1$. Solving $f(x,0) = 0$ and $f(x,-1) = 0$ gives points $(0,0), (3/2,0), (1,-1), (-1/2,-1)$ in Z. At all other points of Z there exists a neighborhood in which the equation $f(x,y) = 0$ can be solved explicitly for y in terms of x by the implicit function theorem. We show at these four points above there is no local explicit solution to $f(x,y) = 0$. Consider $(3/2, 0)$: for $x > 3/2$ there is no y close to 0 with $f(x,y) = 0$ since $2y^3 + 3y^2$ has local minimum 0 when $y = 0$ and $2x^3 - 3x^2 > 0$ here. A similar argument works for $(-1/2, -1)$. The treatment of (0,0) and (1,-1) are similar so we just consider (0,0). For x close to 0 there are two values y_1, y_2 close to 0 with $f(x, y_1) = f(x, y_2) = 0$ (look at the graphs of $2x^3 - 3x^2$ and of $2y^3 + 3y^2$). Hence there is no g with $[y = g(x)$ iff $f(x,y) = 0]$ (close to (0,0)).

(4) Let $f(x,y,u,v,w) = (u^5 + xv^2 - y + w, v^5 + yu^2 - x + w, w^4 + y^5 - x^4)$, and we want to solve for (u,v,w) in terms of (x,y), nearby the point $(x,y,u,v,w) = (1,1,1,1,-1)$. If $\vec{z} = (u,v,w)$ then $f'_{\vec{z}} = \begin{bmatrix} 5u^4 & 2xv & 1 \\ 2yu & 5v^4 & 1 \\ 0 & 0 & 4w^3 \end{bmatrix}$, which equals $\begin{bmatrix} 5 & 2 & 1 \\ 2 & 5 & 1 \\ 0 & 0 & -4 \end{bmatrix}$ at $(x,y,u,v,w) = (1,1,1,1,-1)$. The determinant of this is $-4(21) \neq 0$, so by the implicit function theorem it is possible to solve for $\vec{z}$ explicitly in terms of x, y, in a neighborhood of the point $(x,y) = (1,1)$, and this solution is continuously differentiable.

(5) Clearly $f(0,1,-1) = 0, f'(0,1,-1) = [1 \ \ 0 \ \ 1]$. Here the implicit function theorem tells us to look at the 'matrix' $A = 1$ (similarly to the first lines of the solution to Q3 above, which is invertible. So by the implicit function theorem there does exist such a differentiable function g. Also by the implicit function theorem $g'(1,-1) = -1^{-1}[0 \ \ 1] = [0 \ \ -1]$.

(6*) This is similar in spirit to the related theorem from the notes. Fix $\vec{x} \in S$ and let $\epsilon > 0$ be given. Choose $\delta > 0$ with $B(\vec{x}, \delta) \subset S$ and $\delta < \frac{\epsilon}{M\sqrt{n}}$. Let $\vec{h} \in \mathbb{R}^n$ with $\|\vec{h}\|_2 < \delta$, and set $v_0, v_1, \cdots, v_n$ be as in that theorem from the notes. Then $|f(\vec{x} + \vec{h}) - f(\vec{x})| = |\sum_{k=1}^m f(\vec{x} + v_k) - f(\vec{x} + v_{k-1})| = |\sum_{k=1}^m \frac{\partial f}{\partial x_k}(\xi_k) h_k|$, after applying the ordinary MVT to $g_j(t) = f(\vec{x} + v_{k-1} + t(v_k - v_{k-1}))$. We obtain $|f(\vec{x} + \vec{h}) - f(\vec{x})| \leq \sum_{k=1}^m M|h_k| \leq M\sqrt{n}\|\vec{h}\|_2 < \epsilon$. Hence f is continuous on S.