Department of Mathematics, University of Houston
Math 4332. Intro to Real Analysis. David Blecher, Spring 2015
Homework 5 Solutions

(1) Note $\sum_{n=1}^{\infty} \frac{n \cdot 1}{n^2+1^2}$ diverges by Calc 2 (e.g. comparison with $\sum_{n=1}^{\infty} \frac{1}{2n}$). So by Theorem 5.1, $\sum_{n,m} \frac{nm}{n^2+m^2}$ diverges. (Or note we are adding infinitely many $a_{n,n} = \frac{1}{2}$).

For the second series, note that $\sum_{n=1}^{\infty} 2^{-n^2}$ converges by Calc 2 (e.g. comparison with $\sum_{n=1}^{\infty} 2^{-n}$). Hence by question (5) below, $\sum_{n,m} 2^{-(n^2+m^2)} = \sum_{n,m} 2^{-n^2} 2^{-m^2}$ converges. [3 points + 3 points]

(2) We prove the first sup in the proof of Theorem 5.1 equals the third sup in the proof. (Similarly the second sup equals the third sup.) For $N, K \in \mathbb{N}$ we have

$$\sum_{n=1}^{N} \sum_{m=1}^{K} a_{m,n} \leq \sum_{n=1}^{N} \sup\{\sum_{m=1}^{M} a_{m,n} : M \in \mathbb{N}\} \leq \sup\{\sum_{n=1}^{N} \sup\{\sum_{m=1}^{M} a_{m,n} : M \in \mathbb{N}\} : N \in \mathbb{N}\},$$

so that

$$\sup\{\sum_{n=1}^{N} \sum_{m=1}^{M} a_{m,n} : \ N, M \in \mathbb{N}\} \leq \sup\{\sum_{n=1}^{N} \sup\{\sum_{m=1}^{M} a_{m,n} : M \in \mathbb{N}\} : N \in \mathbb{N}\}.$$

For the reverse inequality, for any $\epsilon > 0$ and $N \in \mathbb{N}$ and $n \in \{1, 2, \cdots, N\}$, we can choose $M_n \in \mathbb{N}$ with $\sum_{m=1}^{M_n} a_{m,n} > \sup\{\sum_{m=1}^{M} a_{m,n} : M \in \mathbb{N}\} - \frac{\epsilon}{N}$. If $M = \max\{M_1, \cdots, M_N\}$ then $\sum_{m=1}^{M} a_{m,n} \geq \sum_{m=1}^{M_n} a_{m,n}$ so

$$\sup\{\sum_{n=1}^{N} \sum_{m=1}^{M} a_{m,n} : \ N, M \in \mathbb{N}\} \geq \sum_{n=1}^{N} \sum_{m=1}^{M} a_{m,n} > \sum_{n=1}^{N} \sup\{\sum_{m=1}^{M} a_{m,n} : M \in \mathbb{N}\} - \epsilon.$$

Letting $\epsilon \to 0$ and taking the sup of the right side over $N \in \mathbb{N}$ gives the desired reverse inequality. [8 points]

(3) This is similar to the proof of the analogous result for ordinary series (second bullet in section 3), and the proof of the result from 3333 that it appeals to. Namely, let $s = \sup\{\sum_{n=1}^{N} \sum_{m=1}^{M} a_{m,n} : N, M \in \mathbb{N}\}$, which is easy to see equals $\sup\{\sum_{n=1}^{N} \sum_{m=1}^{N} a_{m,n} : N \in \mathbb{N}\}$. Then $s < \infty$ iff for any $\epsilon > 0$ there exists an $K \in \mathbb{N}$ with $\sum_{n=1}^{K} \sum_{m=1}^{K} a_{m,n} > s - \epsilon$. This is equivalent to saying that $\sum_{n=1}^{N} \sum_{m=1}^{M} a_{m,n} > s - \epsilon$ whenever $N \geq K$ and $M \geq K$. But saying that $\sum_{n=1}^{N} \sum_{m=1}^{M} a_{m,n} > s - \epsilon$ is the same as saying $|s - \sum_{n=1}^{N} \sum_{m=1}^{M} a_{m,n}| < \epsilon$. Thus the partial sums of $\sum_{n,m} a_{m,n}$ are bounded above iff the definition of $\sum_{n,m} a_{m,n}$ converging (to s) holds. [5 points]

(4) We omit this proof since there are many ways to do it.

(5) Suppose that $\sum_n a_n$ and $\sum_n b_n$ are absolutely convergent, with sums s and t respectively. By Theorem 5.1,

$$\sum_{n,m} |a_m \, b_n| = \sum_n (\sum_m |a_m| \, |b_n|) = \sum_n |b_n| (\sum_m |a_m|) = (\sum_m |a_m|)(\sum_n |b_n|) < \infty.$$

So $\sum_{n,m} a_n \, b_m$ is an absolutely convergent double series. By Theorem 5.2 its sum is [6 points]

$$\sum_{n,m} a_m \, b_n = \sum_n (\sum_m a_m \, b_n) = \sum_n b_n (\sum_m a_m) = (\sum_m a_m)(\sum_n b_n) = st.$$

(6) The point is that by Question 5 above, $\sum_{n,m} a_n \, b_m$ is absolutely convergent. By Theorem 5.3 and its proof, its sum, which is st, is the same as the sum of the absolutely convergent ordinary series

$$a_0 b_0 + a_0 b_1 + a_1 b_0 + a_0 b_2 + a_1 b_1 + a_2 b_0 + \cdots .$$

By adding parentheses and using Homework 4 Question 1, the latter has the same sum as the series

$$a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \cdots ,$$

which is exactly the Cauchy product series. This is absolutely convergent since

$$|a_0 b_0| + |a_0 b_1 + a_1 b_0| + |a_0 b_2 + a_1 b_1 + a_2 b_0| + \cdots \leq |a_0 b_0| + |a_0 b_1| + |a_1 b_0| + |a_0 b_2| + \cdots ,$$

which we said was finite. [2 points for completeness only]

(7) Let $t_n = s_n - s, \tau_n = \sigma_n - s$. Then it is easy to see that $\tau_n = \frac{1}{n}(t_1 + t_2 + \cdots + t_n)$. Let $\epsilon > 0$ be given. Since $t_n \to 0$ the sequence is bounded: $|t_n| \leq K$ for all n, say, and there is an N with $|t_n| \leq \epsilon$ for $n \geq N$. Then

$$|\tau_n| \leq \frac{1}{n}(|t_1| + |t_2| + \cdots + |t_N|) + \frac{1}{n}(|t_{N+1}| + \cdots + |t_n|) < \frac{NK}{n} + \epsilon, \qquad n \geq N.$$

From this it is clear that $\tau_n = \sigma_n - s \to 0$. So $\sigma_n \to s$.