Department of Mathematics, University of Houston
Math 4332. Intro to Real Analysis. David Blecher, Spring 2015
Homework 7

(1) Show that if $f_n \to f$ uniformly on X then $f_n \to f$ pointwise.

(2) Notes Problems 1.8 [3 points], 1.9 [1 point], 1.23 [3 points for completion only].

(3) Let $f_n(x) = n^c x(1-x^2)^n$ for $x \in [0,1]$. Prove that $f_n \to 0$ pointwise on $[0,1]$ for every real c. For which c does $f_n \to 0$ uniformly on $[0,1]$? For which values of c is $\int_0^1 (\lim_{n\to\infty} f_n(x))\,dx = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx$? [1+8+3 points]

(4) Suppose that $f_n \to f$ uniformly on X. If each f_n is bounded, show that f is bounded, and show that (f_n) is uniformly bounded, that is, there is a constant $M > 0$ with $|f_n(x)| \le M$ for all $n \in \mathbb{N}$ and $x \in X$. [2+4 points]

(5) Show that $B(X) = \{f : X \to \mathbb{R}, f \text{ bounded}\}$ is a metric space with metric $d(f,g) = \|f-g\|_\infty = \sup\{|f(x)-g(x)| : x \in X\}$. Show that $B(X)$ is complete. Show that if $f_n \to f$ uniformly and $g_n \to g$ uniformly then $f_n + g_n \to f + g$ uniformly on X and $f_n \cdot g_n \to f \cdot g$ uniformly on X. [2 + 2 + 3 points]

(6) If $g_n \to g$ uniformly on (a,b), and h is a continuous bounded function on (a,b), and if (c_n) is a sequence of constants with limit c, show that $c_n + h(x)\,g_n(x) \to c + h(x)\,g(x)$ uniformly on (a,b). [2 points completion only]

(7) Prove the Cauchy criterion for uniform convergence on X; namely that a sequence (f_n) of scalar valued functions on a set X converges uniformly iff for any $\epsilon > 0$ there exists N such that

$$|f_m(x) - f_n(x)| \le \epsilon, \qquad x \in X, m \ge N, n \ge N.$$

(The latter is saying $\|f_m - f_n\|_\infty \le \epsilon$ whenever $m \ge N, n \ge N$.) [3 points completion only]

(8) Let $f_n(x) = \frac{1}{n} e^{-n^2 x^2}$ for $x \in \mathbb{R}$. Prove that $f_n \to 0$ uniformly on $\mathbb{R}$, that $f_n' \to 0$ pointwise on $\mathbb{R}$, but $f_n' \not\to 0$ uniformly on any interval containing 0. [2 + 2 + 3 points]

(9) Let $\{f_n\}$ be a sequence of real valued bounded functions on $[0,1]$, and $f_n \to f$ uniformly on $[0,1]$. Prove that $\lim_{n\to\infty} \int_0^{1-1/n} f_n(x)\,dx = \int_0^1 f(x)\,dx$. [4 points completion only]