Department of Mathematics, University of Houston
Math 4332. Intro to Real Analysis. David Blecher, Spring 2015
Homework 7 Key

(1) See e.g. page 1 of the notes for this chapter, or note $|f_n(x) - f(x)| \leq \|f_n - f\|_\infty \to 0$, for each x.

(2) Notes Problem 1.8: This is similar to Example 1.4 in the notes: By Calculus 2, $x^{\frac{1}{n}} \to 1$ if $x \in (0, 1]$. Then $s_n = \|f_n - f\|_\infty \geq \sup\{|x^{\frac{1}{n}} - 1| : x \in (0, 1]\} = 1 \nrightarrow 0$. So $f_n \nrightarrow f$ uniformly on $[0, 1]$. [3 points]

1.9: Follows immediately from Dini. [1 point]

1.23: Recall that if $\vec{a} = (a_i) \in \mathbb{R}^k$ then

$$|a_j| \leq \|\vec{a}\| = \sqrt{\sum_{i=1}^{k} |a_i|^2} \leq \sqrt{n} \max_i |a_i|.$$

Hence

$$|f_{j,n}(x) - g_j(x)| \leq \|f_n(x) - g(x)\| \leq \sqrt{n} \max_i \{|f_{i,n}(x) - g_i(x)|\}, \qquad x \in X,$$

so that in the 'γ' notation of 1.20/1.21 in the notes,

$$\|f_{j,n} - g_j\|_\infty \leq \gamma(f_n, g) \leq \sqrt{n} \max_i \|f_{i,n} - g_i\|_\infty.$$

Thus $\gamma(f_n, g) \to 0$ as $n \to \infty$ iff $\|f_{i,n} - g_i\|_\infty \to 0$ as $n \to \infty$ for every $i = 1, \cdots, k$. This is saying $f_n \to g$ uniformly iff $f_{i,n} \to g_i$ uniformly for every $i = 1, \cdots, k$. [3 points for completion only]

(3) Here $f_n(x) = n^c x(1 - x^2)^n \to 0$ pointwise on $[0, 1]$ since by Calculus, $n^c \delta^n \to 0$ for $\delta \in [0, 1)$. By the Calculus I technique, f_n has a maximum value of $\frac{n^c}{\sqrt{2n+1}}(\frac{2n}{2n+1})^n$ (achieved when $x = \frac{1}{\sqrt{2n+1}}$). Hence if $c < \frac{1}{2}$ then $\|f_n - 0\|_\infty \leq \frac{n^c}{\sqrt{2n+1}} \to 0$ as $n \to \infty$, so $f_n \to 0$ uniformly on $[0, 1]$. If $c \geq \frac{1}{2}$ then $\|f_n - 0\|_\infty \geq \frac{\sqrt{n}}{\sqrt{2n+1}}(\frac{2n}{2n+1})^n \nrightarrow 0$. So $f_n \nrightarrow f$ uniformly on $[0, 1]$. Finally, $\int_0^1 f_n(x)\,dx = \frac{n^c}{2(n+1)} \to \int_0^1 f\,dx = 0$ iff $c < 1$. [1+8+3 points]

(4) We use notation from Q 5 below. If each f_n is bounded then f is bounded since

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \leq \|f - f_n\|_\infty + \|f_n\|_\infty.$$

Suppose that $\|f_n - f\|_\infty < 1$ for $n \geq N$. Since $f_n = (f_n - f) + f$, by the triangle inequality we have

$$\|f_n\|_\infty \leq \|f_n - f\|_\infty + \|f\|_\infty < 1 + \|f\|_\infty, \qquad n \geq N.$$

So $M = \max\{\|f_1\|_\infty, \cdots, \|f_N\|_\infty, 1 + \|f\|_\infty\}$ will work. [2+4 points]

(5) That $d(f, g) = \|f - g\|_\infty = \sup\{|f(x) - g(x)| : x \in X\}$ is a complete metric is the same as the proof in 1.21 in the notes, except for why $d(f, g) < \infty$. The latter is because $|f(x) - g(x)| \leq |f(x)| + |g(x)| \leq K + M$ if K and M are upper bounds for $|f|$ and $|g|$. If $f_n \to f$ uniformly and $g_n \to g$ uniformly then by the triangle inequality

$$\|f_n + g_n - (f + g)\|_\infty \leq \|f_n - f\|_\infty + \|g_n - g\|_\infty \to 0.$$

So $f_n + g_n \to f + g$ uniformly on X. Also, writing $f_n \cdot g_n - f \cdot g = f_n \cdot g_n - f \cdot g_n + f \cdot g_n - f \cdot g$, and using Question 4 to get upper bounds K and M for $|g_n|$ and $|f|$ respectively, we have

$$|f_n(x)g_n(x) - f(x)g(x)| \leq |f_n(x) - f(x)||g_n(x)| + |f(x)||g_n(x) - g(x)| \leq K\|f_n - f\|_\infty + M\|g_n - g\|_\infty.$$

Thus $\|f_n \cdot g_n - f \cdot g\|_\infty \leq K\|f_n - f\|_\infty + M\|g_n - g\|_\infty \to 0$, and so $f_n \cdot g_n \to f \cdot g$ uniformly on X. [2 + 2 + 3 points]

(6) This follows as in the last part of Q 5 above: If M is an upper bound for $|h|$ then [2 points completion only]

$$|c_n + h(x)\,g_n(x) - (c + h(x)\,g(x))| \leq |c_n - c| + |h(x)g_n(x) - h(x)\,g(x)| = |c_n - c| + |h(x)||g_n(x) - g(x)| \leq |c_n - c| + M\|g_n - g\|_\infty.$$

(7) ($\Leftarrow$) This is exactly as in the last few lines of the proof of 1.21.

($\Rightarrow$) If $f \overset{u}{\to} f$ uniformly then given $\epsilon > 0$ there exists N such that

$$|f_n(x) - f(x)| \leq \frac{\epsilon}{2}, \qquad x \in X, n \geq N.$$

Hence [3 points completion only]

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \qquad x \in X, m, n \geq N.$$

(8) Note that $\|f_n - 0\|_\infty \leq \frac{1}{n} \to 0$, so $f_n \to 0$ uniformly. Also, $f_n'(x) = -2nxe^{-n^2x^2} \to 0$ for all $x \in \mathbb{R}$ since $\lim_{n\to\infty} ne^{-n^2K} = 0$ by Calculus, if $K > 0$. However, $f_n'(\frac{1}{n}) = -2e^{-1}$, so $\|f_n' - 0\|_\infty \geq 2e^{-1}$. Hence $f_n' \not\to 0$ uniformly (not even on any interval containing 0). [2 + 2 + 3 points]

(9) Let M be as in Q 4 above, then [4 points completion only]

$$|\int_0^{1-1/n} f_n(x)\,dx - \int_0^1 f_n(x)\,dx| = |\int_{1-1/n}^1 f_n(x)\,dx| \leq \int_{1-1/n}^1 |f_n(x)|dx \leq \int_{1-1/n}^1 M dx = \frac{M}{n} \to 0,$$

as $n \to \infty$. Also, by 1.14, $\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx$. So by the triangle inequality,

$$|\int_0^{1-1/n} f_n(x)\,dx - \int_0^1 f(x)\,dx| \leq |\int_0^{1-1/n} f_n(x)\,dx - \int_0^1 f_n(x)\,dx| + |\int_0^1 (f_n(x) - f(x))\,dx| \to 0.$$