Department of Mathematics, University of Houston
Math 4332. Intro to Real Analysis. David Blecher, Spring 2015
Homework 8 Key

(1) This is like the proof of the Alternating Series Test from Chapter 0, so follow along with that argument: Note that $g_{n+k} - g_{n+k+1} \geq 0$, so $g_n - g_{n+1} + g_{n+2} - g_{n+3} + \cdots \geq 0$. If $m > n$ then

$$|s_m - s_n| = g_n - g_{n+1} + g_{n+2} - \cdots = g_n - (g_{n+1} - g_{n+2}) - (g_{n+3} - g_{n+4}) - \cdots \leq g_n,$$

since $g_{n+k+1} \leq g_{n+k}$. Thus $\|s_m - s_n\|_\infty \leq \|g_n\|_\infty \to 0$ as $n \to \infty$. It follows that (s_n) is Cauchy, so convergent. That is, $\sum_{k=1}^\infty (-1)^{k+1} g_k$ converges uniformly. [3 points for completion only]

(2*) Suppose $\gamma : [0,1] \to [0,1] \times [0,1]$ is continuous, one-to-one, and onto. Then $g = \gamma^{-1}$ is well defined. If $E \subset [0,1]$ is closed then E is compact, so that $\gamma(E)$ is compact by 3.37 in Math 4331. Hence $g^{-1}(E)$ is closed. So g is continuous by the characterization of continuous functions in 4331. Now $[0,1] \times [0,1] \setminus \{\gamma(\frac{1}{2})\}$ is connected, so that $g([0,1] \times [0,1] \setminus \{\gamma(\frac{1}{2})\}) = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ is connected (by a result on connectedness in 4331), a contradiction. [5 points for grad students only]

(3) $\frac{1}{n^2} 2^{-x^2/n} \leq \frac{1}{n^2}$, and $\sum_n \frac{1}{n^2} < \infty$ so $\sum_{n=1}^\infty \frac{1}{n^2} 2^{-x^2/n}$ converges uniformly by the Weierstrass M-test, and is continuous by the theorem in class on continuity of infinite series. Problem 1.30 in the notes is similar, but notice by the Calculus I technique, f_n has a maximum value of $\frac{1}{2n^{\frac{3}{2}}}$ (achieved when $x = \frac{1}{n^{\frac{3}{2}}}$). Thus $|f_n| \leq \frac{1}{2n^{\frac{3}{2}}}$, and $\sum_n \frac{1}{2n^{\frac{3}{2}}} < \infty$. The radius of convergence of $\sum_{n=1}^\infty \frac{x^n}{n}$ is 1 so this is continuous on $(-1, 1)$. [3+2+2 points]

(4) When $x = 0$ this diverges. If $x \neq 0$ then by the limit comparison test it converges (compare with $\sum_{n=1}^\infty \frac{1}{n^2}$). If $r > 0$ then $\sum_{n=1}^\infty \frac{1}{1+n^2x^2}$ converges uniformly on $(-\infty, -r] \cup [r, \infty)$ by the Weierstrass M-test since $\frac{1}{1+n^2x^2} \leq \frac{1}{1+n^2r^2}$, and $\sum_{n=1}^\infty \frac{1}{1+n^2r^2}$ converges as we said above. So by a theorem in class on continuity of infinite series, $f(x)$ is continuous on $(-\infty, -r] \cup [r, \infty)$, for all $r > 0$, hence on $(-\infty, 0) \cup (0, \infty)$. So f is continuous whenever the series converges. If $1 \leq n \leq m$ and $x = \pm\frac{1}{m}$ then $\frac{1}{1+n^2x^2} \geq \frac{1}{1+m^2x^2} = \frac{1}{2}$, so that

$$f(x) \geq \sum_{n=1}^m \frac{1}{1+n^2x^2} \geq \sum_{n=1}^m \frac{1}{2} = m/2.$$

So f is unbounded on $(-\infty, 0) \cup (0, \infty)$, and on any interval I with endpoint 0. By Homework 7 Q 4, the series cannot converge uniformly on such an interval I. [3+4+1+4 points]

(5) (a) 1, since $\lim_n n^{\frac{1}{2n}} = 1$. (b) 1, since $\limsup_n |a_n|^{\frac{1}{n}} = 1$ here. (c) ∞, since $\lim_n (\frac{1}{n^n})^{\frac{1}{n}} = \lim_n \frac{1}{n} = 0$. [1+2+1 points]

(6) $\limsup_n |a_n|^{\frac{1}{n}} = \frac{1}{2}$ here, so (a) $R = \frac{1}{\limsup_n |a_n^3|^{\frac{1}{n}}} = 2^3 = 8$. (b) $R = \frac{1}{\limsup_n |a_n|^{\frac{1}{3n}}} = 2^{\frac{1}{3}}$. (c) $R = \frac{1}{\limsup_n |a_n|^{\frac{1}{n^2}}} = 1$. [2+2+2 points]

(7*) If $|z - z_1| < r - |z_0 - z_1|$ then $|z - z_0| < r$, so that $f(z) = \sum_{n=0}^\infty a_n (z - z_0)^n$ converges absolutely. Also, $\sum_{n=0}^\infty (\sum_{k=0}^n \binom{n}{k} |z - z_1|^k |z_1 - z_0|^{n-k}) = \sum_{n=0}^\infty (|z - z_1| + |z_1 - z_0|)^n$ converges since $|z - z_1| + |z_1 - z_0| < r$, hence

$$f(z) = \sum_{n=0}^\infty a_n (z - z_0)^n = \sum_{n=0}^\infty a_n (z - z_1 + z_1 - z_0)^n = \sum_{n=0}^\infty a_n \sum_{k=0}^n \binom{n}{k} (z - z_1)^k (z_1 - z_0)^{n-k} = \sum_{k=0}^\infty (\sum_{n=k}^\infty \binom{n}{k} a_n (z_1 - z_0)^{n-k})(z - z_1)^k,$$

since the latter may be viewed by Theorem 5.2 in Chapter 0 as an absolutely convergent double series which may be rewritten by Theorem 5.3 in Chapter 0 as the ordinary series $\sum_{n=0}^\infty a_n \sum_{k=0}^n \binom{n}{k} (z - z_1)^k (z_1 - z_0)^{n-k}$. [5 points grad students only]

(8*) Let A be the set of limit points of E in $B(0, R)$, and let $B = B(0, R) \setminus A$. If $x \in B$ then there exists $\epsilon > 0$ with $B(x, \epsilon) \cap E \subset \{x\}$. Any point in $B(x, \epsilon)$ is in B, so B is open. If $x \in A$ then by Question 7 we may write $f(z) = \sum_{n=0}^\infty (a_n - b_n) z^n = \sum_{n=0}^\infty d_n (z - x)^n$ valid if $|z - x| < R - |x|$. We claim $d_n = 0$ for all n. Otherwise let $k = \min\{j : d_j \neq 0\}$. Then $f(z) = (z - x)^k g(z)$ where $g(z) = \sum_{m=0}^\infty d_{k+m} (z - x)^m$. Now g is continuous at x and $g(x) \neq 0$ so there is a $\delta > 0$ with g never zero on $B(x, \delta)$. So f is never zero on $B(x, \delta)$ except at x, contradicting that x is a limit point of E. So $d_n = 0$ for all n, so $f = 0$ on $B(x, R - |x|)$, and so A is open. Since A is nonempty and $B(0, R)$ is connected we see $B = \emptyset$. Since f is continuous on $B(0, R)$ and zero on E, it is also zero on A. This implies that $A \subset E$, so $E = B(0, R)$. So $a_n = b_n$ for all n by a corollary to the theorem on differentiation on power series. [3 points for completeness grad students only]

(9) Note $|a_k x^k| \leq |a_k|$ and $\sum_k |a_k| < \infty$, so Problem 1.51 follows from the Weierstrass M-test. For Problem 1.66, consider $\sum_{n=0}^{\infty} a_n R_n z^n$, which has radius of convergence $\frac{1}{\limsup_n |a_n R^n|^{\frac{1}{n}}} = \frac{1}{\limsup_n |a_n|^{\frac{1}{n}} R} = 1$. So the case we did prove, we have $\lim_{z \to 1^-} \sum_{n=0}^{\infty} a_n R^n z^n = \sum_{n=0}^{\infty} a_n R^n$. Letting $x = Rz$, or $z = x/R$, we deduce that $\lim_{x \to R^-} \sum_{n=0}^{\infty} a_n x^n = \lim_{z \to 1^-} \sum_{n=0}^{\infty} a_n R^n z^n = \sum_{n=0}^{\infty} a_n R^n$. [3+6 points]