Department of Mathematics, University of Houston
Math 4332. Intro to Real Analysis. David Blecher, Spring 2015
Homework 9 Key.

(1) Using the definitions from class,

$$e^{iz} = 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - \cdots = (1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots) + i(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots) = \cos z + i \sin z.$$

If $\cos x = 0$ has no real solution then (or equivalently, since $\cos(-x) = \cos x$, it has no positive solution), then by the IVT cos is always strictly positive. So sin is always strictly increasing, and since $\sin 0 = 0$ we have that sin is always strictly positive. Fix $x > 0$. Then

$$(y - x) \sin x \leq \int_x^y \sin t \, dt = \cos x - \cos y \leq 2, \qquad y > x,$$

since we saw $\cos^2 + \sin^2 = 1$. So $y \leq \frac{2}{\sin x} + x$, so the latter constant is an upper bound for all $y \in \mathbb{R}$. This is a contradiction so $\cos x = 0$ has a positive solution.

Since $\cos^2 + \sin^2 = 1$, and we saw cos decreases and sin increases strictly on $[0, \frac{\pi}{2}]$, and $e^{i\frac{\pi}{2}} = i$, on that subinterval γ traces out the first quadrant of the unit circle counterclockwise in a one-to-one fashion. On $[\frac{\pi}{2}, \pi]$ we can use the fact that

$$e^{i(\theta + \frac{\pi}{2})} = e^{i\theta} e^{i\frac{\pi}{2}} = ie^{i\theta} = (-\sin\theta, \cos\theta), \qquad \theta \in [0, \frac{\pi}{2}],$$

to see that γ traces out the second quadrant of the unit circle on $[\frac{\pi}{2}, \pi]$, counterclockwise and in a one-to-one fashion. Similarly, since

$$e^{i(\theta + \pi)} = e^{i\theta} e^{i\pi} = -e^{i\theta} = (-\cos\theta, -\sin\theta), \qquad \theta \in [0, \pi],$$

we see that γ traces out the bottom half of the unit circle on $[\pi, 2\pi]$, counterclockwise and in a one-to-one fashion. So γ is one-to-one on $[0, 2\pi)$. If $\gamma([0, 2\pi))$ omitted a point on the circle then it would be disconnected, which by a Theorem from 4331 would give the contradiction that $[0, 2\pi)$ is disconnected. [2+3+3+2 points]

(2) By the Weierstrass polynomial approximation theorem, there is a sequence of polynomials r_n with $r_n \xrightarrow{u} f$ on $[a, b]$. Then $r_n(a) \to f(a) = 0$ so $p_n = r_n - r_n(a) \xrightarrow{u} f$ on $[a, b]$. [3 points]

(3) Problem 1.80: If $M = 0$ then $f = 0$ and this is obvious. So suppose that $M > 0$. Let r_n be as above, then $\|r_n\|_\infty \to \|f\|_\infty = M$ (if you like, because the norm is continuous: by the triangle inequality $|\|r_n\|_\infty - \|f\|_\infty| \leq \|r_n - f\|_\infty \to 0$). So $p_n = Mr_n / \|r_n\|_\infty \xrightarrow{u} f$ and $\|p_n\|_\infty = M$.

Problem 1.82*: This should probably be 'starred'. As suggested we do this by induction on K. If $K = 1$, let r_n be as above, then $r_n(x_1) \to f(x_1)$ so $p_n = r_n - r_n(x_1) + f(x_1) \xrightarrow{u} f$ and $p_n(x_1) = f(x_1)$. Assume we have found a sequence q_n of polynomials with $q_n(x_i) = f(x_i)$ for all $i = 1, \cdots, K-1$, and $q_n \xrightarrow{u} f$. Let $h(x) = (x - x_1)(x - x_2) \cdots (x - x_{K-1})$, and set $p_n = q_n + (f(x_K) - q_n(x_K))h(x)/h(x_K)$. This does the job (check it).

(4) We show that $f * g$ is uniformly continuous (grad students only). As we said in class, $\|f\|_\infty \leq M$ and f is uniformly continuous on the compact interval it is supported on, and similarly for g. That is, given $\epsilon > 0$ there is a $\delta > 0$ with $|g(s) - g(t)| \leq \epsilon/K$ whenever $|s - t| < \delta$. The constant K is chosen to equal $\int |f(y)| dy$ (note this is really an integral of a continuous function on a bounded interval containing the support of f, so is finite). So if $|s - t| < \delta$ then

$$|(f * g)(s) - (f * g)(t)| = |\int f(y)g(s - y)dy - \int f(y)g(t - y)dy| = |\int f(y)(g(s - y) - g(t - y))dy|,$$

which is, since $|(s - y) - (t - y)| = |s - t| < \delta$, is dominated by

$$\int |f(y)||g(s - y) - g(t - y)|dy \leq \int |f(y)| \frac{\epsilon}{K} dy = \epsilon.$$

So $f * g$ is uniformly continuous. (grad students only).

Then [3 points:]

$$(f * (g + ch))(x) = \int f(t)(g + ch)(x - t)dt = \int f(t)g(x - t)dt + c\int f(t)h(x - t)dt = (f * g + c(f * h))(x).$$

(5) This was done in class–I just want you to say it in your own words.

(6) Problem 1.89: $\mathcal{A}$ satisfies all the conditions of the Stone-Weierstrass theorem. For example, we show that $\mathcal{A}$ separates points of K: if $\vec{z} \neq \vec{y}$ in K, then for some i we have $z_i \neq y_i$, so the polynomial $\vec{x} \mapsto x_i$ takes different values at these two points. So $\mathcal{A}$ is dense.

The Stone-Weierstrass theorem implies the Weierstrass polynomial approximation theorem by taking $\mathcal{A}$ to be the polonomials–these satisfy all the conditions of the Stone-Weierstrass theorem, so they are dense. [5+2 points]