Department of Mathematics, University of Houston
4332 - Intro to Real Analysis Second Semester - Blecher
Test 1–Spring 2015

Instructions: Put all your bags and papers on the side of the room. Answer question 0 [40 points] and any one question from questions 1–3 [25 points], and any one question from questions 4–7 [50 points]. Besides these, do not attempt parts of other questions (for example if you attempt parts of 0, 2, 4, 6, 7, only 0,2,6 will be graded. SHOW ALL YOUR REASONING. Time: 85 minutes. You may quote freely any results from the notes without proof, except those you are asked to prove. Possible total [115 points].

0. (a) If $\sum_{k=1}^{\infty} a_k$ converges, and $\sum_{k=1}^{\infty} b_k$ is obtained from $\sum_{k=1}^{\infty} a_k$ by adding parentheses, show that $\sum_{k=1}^{\infty} b_k$ converges. What is its sum? [8 points]
   (b) Does $\sum_{n,m=1}^{\infty} \frac{1}{n^2+m^2}$ converge? [Hint: $\sum_{m=1}^{\infty} \frac{1}{n^2+m^2} \geq \sum_{m=1}^{n} \frac{1}{n^2+m^2}$, and $\frac{1}{n^2+m^2} \geq \frac{1}{2n^2}$ if $m \leq n$.] [6 points]
   (c) State Dini's theorem, and use it to show that $g_n(x) = (e^{\frac{x}{n}})$ converges uniformly on $[0, 1]$. [10 points]
   (d) Show that if $\sum_{k=1}^{\infty} a_k$ converges absolutely then $\sum_{k=1}^{\infty} a_k x^k$ converges uniformly to a continuous function on $[-1, 1]$. [6 points]
   (e) Let $f(x) = \sum_{k=1}^{\infty} \frac{e^{kx}}{k^k}$, for all x where this series converges. Find where f is continuous, where it is differentiable, and calculate $f'(x)$. [10 points]

1. (a) Prove the Cauchy criterion for convergence of a series of real numbers. [12 points]
   (b) Prove that if $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge then $\sum_{k=1}^{\infty}(a_k + b_k)$ converges. [5 points]
   (c) What is the 'tail' of a series? Show that if a series converges then its tail converges. [5 points]
   (d) Complete the sentence: "A nonnegative series converges iff the sequence ________________ is ________________ above, and then the sum of the series equals the ______________". [3 points]

2. (a) (This item was deleted on actual test.) If $a_{nm} \geq 0$ for all $n, m \in N$, show that $\sum_{n,m=1}^{\infty} a_{nm}$ converges if and only if the set how that if $\sum_{k=1}^{\infty} a_k$ converges absolutely then $\sum_{k=1}^{\infty} a_k x^k$ converges ints] uniformly to a continuous function on $[-1, 1]$. [6 points]
   then the sum equals the supremum of this set.
   (b) What is the *Cauchy product* of two series of numbers?
   (c) State the theorem about writing double series as ordinary series.

(d) Prove that the Cauchy product of two absolutely convergent series is convergent. [3+7+15 points]

3. (a) Show that if $s_n \geq 0$ and $t_n \geq 0$ for all n, then $\text{limsup}_n\,(s_n t_n) \leq (\text{limsup}_n\, s_n)(\text{limsup}_n\, t_n)$, and $\text{limsup}_n\,(s_n t_n) = (\text{limsup}_n\, s_n)(\text{limsup}_n\, t_n)$ if (t_n) converges to a number $t > 0$.
   (b) What is a 'rearrangement' of an infinite series? Fill in the blanks: "Any 'rearrangement' of an absolutely convergent series is ______________________________ sum."
   (c) State the condensation test.
   (d) Use the condensation test to check if $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^3}$ converges. [9+6+6+4 points]

4. (a) Define *uniform convergence of a sequence of real valued functions* on a subset S of $\mathbb{R}$.
   (b) Show that if $f_n \to f$ uniformly on (a, b), and if each f_n is continuous on (a, b), then f is continuous on (a, b).
   (c) Show that $(\frac{x}{n} e^{-\frac{x}{n}})$ does not converge uniformly on $[0, \infty)$.
   (d) State the Cauchy condition for uniform convergence of a sequence of functions. [5+26+11+8 points]

5. (a) Define *uniform convergence of a series of real valued functions* on a subset S of $\mathbb{R}$.
   (b) State and prove the Weierstrass M-test.
   (c) Show that $\sum_{n=1}^{\infty} \frac{x}{1+n^4x^2}$ converges and is continuous on $\mathbb{R}$.
   (d) State conditions ensuring that if $f_n \to f$ then f is differentiable and $\lim_{n\to\infty} f_n' = f'$ pointwise. [6+22+9+8 points]

6. (a) Define the radius of convergence of a power series.
   (b) Find the radius of convergence of $\sum_{k=1}^{\infty} 3^k x^{2k}$.
   (c) State and prove the main theorem about the derivative of a power series $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ with positive radius of convergence r.
   (d) Find where the series $\sum_{n=1}^{\infty} nx^n$ is differentiable, and give a formula for the derivative here. [6+10+24+10 points]

7. (a) Define the convolution $f*g$ of two continuous scalar functions on $\mathbb{R}$ of compact support, and explain why the integral here exists.
   (b) Prove that $f * (g + h) = f * g + f * h$.
   (c) State the Stone-Weierstrass theorem (real not complex case). Then give two other equivalent formulations of it and say why they are equivalent.

(d) Let K be a compact set in $\mathbb{R}^n$. Show that the polynomials in n variables are dense in the set of continuous scalar functions on K. [12+5+20+14 points]