Department of Mathematics, University of Houston
4332 - Intro to Real Analysis Second Semester - Blecher
Test 1 Key–Spring 2015

Intstructions: Put all your bags and papers on the side of the room. Answer question 0 [40 points] and any one question from questions 1–3 [25 points], and any one question from questions 4–7 [50 points]. Besides these, do not attempt parts of other questions (for example if you attempt parts of 0, 2, 4, 6, 7, only 0,2,6 will be graded. SHOW ALL YOUR REASONING. Time: 85 minutes. You may quote freely any results from the notes without proof, except those you are asked to prove. Possible total [115 points]

0. (a) Suppose that $1 \leq n_1 < n_2 < \cdots$ are integers, and $b_1 = \sum_{k=1}^{n_1} a_k, b_2 = \sum_{k=n_1+1}^{n_2} a_k, b_3 = \sum_{k=n_2+1}^{n_3} a_k, \cdots$. Let $s_n = \sum_{k=1}^{n} a_k$. Note that $b_1 = s_{n_1}, b_1 + b_2 = s_{n_2}, \cdots$, generally $b_1 + b_2 + \cdots + b_N = s_{n_N}$. So the partial sums of $\sum_{k=1}^{\infty} b_k$ converge to $\sum_{k=1}^{\infty} a_k$, since they are a subsequence of (s_n), and subsequences of convergent sequences converge with the same limit. So $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} b_k$. [8 points]

   (b) No, because by the Hint, $\sum_{m=1}^{\infty} \frac{1}{n^2+m^2} \geq \sum_{m=1}^{n} \frac{1}{n^2+m^2} \geq \sum_{m=1}^{n} \frac{1}{2n^2} = \frac{1}{2n}$, so [6 points]

$$\sum_{n,m=1}^{\infty} \frac{1}{n^2+m^2} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{n^2+m^2} \geq \sum_{n=1}^{\infty} \frac{1}{2n} = +\infty.$$

   (c) Dini's theorem states that if X is a compact metric space and continuous real valued functions $f_n \to f$ pointwise on X, and $f_1 \geq f_2 \geq f_3 \geq \cdots$, and f is continuous, then $f_n \to f$ converges uniformly on X.

   Note that $g_n(x) = e^{\frac{x}{n}} \to e^0 = 1$ pointwise on $[0,1]$ as $n \to \infty$. Also all these functions are continuous and $g_1 \geq g_2 \geq \cdots$, so by Dini $g_n \to 1$ uniformly on $[0,1]$. [10 points]

   (d) Note $|a_k x^k| \leq |a_k|$ and $\sum_k |a_k| < \infty$, so by the Weierstrass M-test $\sum_{k=1}^{\infty} a_k x^k$ converges uniformly on $[-1,1]$. Since each $a_k x^k$ is continuous, by the theorem on continuity of series $\sum_{k=1}^{\infty} a_k x^k$ is a continuous function on $[-1,1]$. [Another proof: by the comparison test the power series converges (absolutely) on $[-1,1]$, so the interval of convergence must contain $[-1,1]$, and from the theory of power series it is continuous on the interval of convergence.] [6 points]

   (e) Short proof: let $g(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^k}$ which is a power series with radius of convergence ∞, so is continuous and differentiable everywhere with derivative $\sum_{k=1}^{\infty} \frac{x^{k-1}}{k^{k-1}}$ by the result about differentiating power series. Note $f(x) = g(e^x)$, so by Calculus f is continuous and differentiable everywhere, and $f'(x) = e^x g'(e^x) = e^x \sum_{k=1}^{\infty} \frac{(e^x)^{k-1}}{k^{k-1}} = \sum_{k=1}^{\infty} \frac{(e^x)^k}{k^{k-1}}$. Longer proof: If $f_k = \frac{e^{kx}}{k^k}$ then $f_k' = \frac{ke^{kx}}{k^k}$. First fix any $r > 0$ and work on the interval $(-\infty, r]$, here $|f_k'| \leq \frac{ke^{kr}}{k^k}$. By the root test $\sum_{k=1}^{\infty} \frac{ke^{kr}}{k^k}$ converges since $(\frac{ke^{kr}}{k^k})^{\frac{1}{k}} = \frac{k^{\frac{1}{k}} e^r}{k} \to 0$. So $\sum_k f_k'$ converges uniformly on $(-\infty, r]$ so by the theorem on differentiation of a series, $\sum_k f_k$ converges uniformly on $(-\infty, r)$ to a (conts) differentiable function $f(x)$ and $f'(x) = \sum_{k=1}^{\infty} \frac{ke^{kr}}{k^k}$. Since this is true for all $r > 0$, f is continuous and differentiable everywhere and its derivative is $f'(x) = \sum_{k=1}^{\infty} \frac{ke^{kr}}{k^k}$. [10 points]

1. (a) The Cauchy test for series says that $\sum_k a_k$ converges iff given $\epsilon > 0$ there exists an $N \geq 0$ such that $|a_{n+1} + a_{n+2} + \cdots + a_m| < \epsilon$ whenever $m > n \geq N$. Proof: Since $s_m - s_n = a_{n+1} + a_{n+2} + \cdots + a_m$, this is just saying that the partial sums $s_n = \sum_{k=1}^{n} a_k$ are a Cauchy sequence. And we know from Math 3333 that a sequence converges iff it is a Cauchy sequence. [12 points]

   (b) The nth partial sum of $\sum_{k=0}^{\infty} (a_k + b_k)$ is $\sum_{k=0}^{n-1} (a_k + b_k) = \sum_{k=0}^{n-1} a_k + \sum_{k=0}^{n-1} b_k$. By a fact about sums of limits of *sequences* from 3333 or 4331, this converges, as $n \to \infty$, to $\sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$. [5 points]

   (c) It is the sequence whose nth term is $\sum_{k=n}^{\infty} a_k$, so it is just the difference between the sum of the series and its partial sum. So if the series converges the partial sum converges to the sum of the series, so this difference converges to 0. [5 points]

   (d) Complete the sentence: "A nonnegative series converges iff the sequence *of partial sums* is *bounded* above, and then the sum of the series equals the *least upper bound*". [3 points]

2. (a) (Deleted from test) Let $s = \sup\{\sum_{n=1}^{N} \sum_{m=1}^{M} a_{m,n} : N, M \in \mathbb{N}\}$, which is easy to see equals $\sup\{\sum_{n=1}^{N} \sum_{m=1}^{N} a_{m,n} : N \in \mathbb{N}\}$. Then $s < \infty$ iff for any $\epsilon > 0$ there exists an $K \in \mathbb{N}$ with $\sum_{n=1}^{K} \sum_{m=1}^{K} a_{m,n} > s - \epsilon$. This is equivalent to saying that $\sum_{n=1}^{N} \sum_{m=1}^{M} a_{m,n} > s - \epsilon$ whenever $N \geq K$ and $M \geq K$. But saying that $\sum_{n=1}^{N} \sum_{m=1}^{M} a_{m,n} > s - \epsilon$ is the same as saying $|s - \sum_{n=1}^{N} \sum_{m=1}^{M} a_{m,n}| < \epsilon$. Thus the partial sums of $\sum_{n,m} a_{m,n}$ are bounded above iff the definition of $\sum_{n,m} a_{m,n}$ converging (to s) holds.

   (b) The Cauchy product of two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is the series $\sum_{n=0}^{\infty} c_n$ where $c_n = \sum_{k=0}^{n} a_k \, b_{n-k}$. [3 points]

   (c) If $\sum_{n,m=1}^{\infty} |a_{m,n}| < \infty$, and $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a bijective function set $b_k = a_{g(k)}$. Then $\sum_k b_k$ converges (absolutely), and equals $\sum_{n,m=1}^{\infty} a_{m,n}$. [7 points]

   (d) By a homework question $\sum_{n,m} a_n \, b_m$ is absolutely convergent. By Theorem 5.3 and its proof, its sum, which is st, is the same as the sum of the absolutely convergent ordinary series

$$a_0 b_0 + a_0 b_1 + a_1 b_0 + a_0 b_2 + a_1 b_1 + a_2 b_0 + \cdots.$$

By adding parentheses and using the result on adding parentheses, the latter has the same sum as the series

$$a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots ,$$

which is exactly the Cauchy product series. This is absolutely convergent since

$$|a_0b_0| + |a_0b_1 + a_1b_0| + |a_0b_2 + a_1b_1 + a_2b_0| + \cdots \leq |a_0b_0| + |a_0b_1| + |a_1b_0| + |a_0b_2| + \cdots ,$$

which we said was finite. [15 points]

3. (a) If $k \geq n$ then $s_k t_k \leq (\sup\{s_k : k \geq n\})(\sup\{t_k : k \geq n\})$, so $\sup\{s_k t_k : k \geq n\} \leq (\sup\{s_k : k \geq n\})(\sup\{t_k : k \geq n\})$. Taking the limit of these as $n \to \infty$ we get $\limsup_n (s_n t_n) \leq (\limsup_n s_n)(\limsup_n t_n)$. Suppose that (t_n) converges to t. (This was not asked for but if $t = 0$ then from the previous lines $0 \leq \limsup_n (s_n t_n) \leq (\limsup_n s_n)(\limsup_n t_n) = 0$, so we get equality.) If $t > 0$, then given $\epsilon > 0$ there exists an $N \geq 1$ such that $t_n > t(1-\epsilon)$ for all $n \geq N$ (if you wish you may use $t - \epsilon$ in place of $t(1-\epsilon)$ here). Then

$$\sup\{s_k t_k : k \geq n\} \geq \sup\{s_k t(1-\epsilon) : k \geq n\} = t(1-\epsilon)\sup\{s_k : k \geq n\}$$

for $n \geq N$. Taking the limit of these as $n \to \infty$ we get $\limsup_n (s_n t_n) \geq (\limsup_n s_n)(\lim_n t_n)(1-\epsilon)$, for all $\epsilon > 0$. So $\limsup_n (s_n t_n) = (\limsup_n s_n)(\limsup_n t_n)$. [9 points]

(b) If $\sum_{k=1}^{\infty} a_k$ is a series, and $f : \mathbb{N} \to \mathbb{N}$ is a bijection, then the series $\sum_{k=1}^{\infty} a_{f(k)}$ is called a 'rearrangement' of $\sum_{k=1}^{\infty} a_k$. Fill in the blanks: "Any 'rearrangement' of an absolutely convergent series is *is convergent and has the same* sum." [6 points]

(c) Suppose that $a_0 \geq a_1 \geq a_2 \geq \cdots \geq 0$, and that $\lim_k a_k = 0$. Then $\sum_k a_k$ converges iff $\sum_k 2^k a_{2^k}$ converges. [6 points]

(d) $\sum_{n=2}^{\infty} 2^n \frac{1}{2^n (n \log 2)^3} = \sum_{n=2}^{\infty} \frac{1}{n^3 (\log 2)^3}$ which converges by the p-series test. So by the condensation test, $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^3}$ converges. [4 points]

4. (a) $f_n \to f$ uniformly on S if $\sup\{|f_n(x) - f(x)| : x \in S\} \to 0$ (or if you like $\|f_n - f\|_\infty \to 0$) as $n \to \infty$. [5 points]

(b) See class notes. [26 points]

(c) Note that $f_n = \frac{x}{n} e^{-\frac{x}{n}}$ converges pointwise to 0. We use the the Calculus I technique: $f_n' = \frac{1}{n} e^{-\frac{x}{n}} - \frac{x}{n^2} e^{-\frac{x}{n}} = 0$ when $x = n$, and $f_n(n) = e^{-1}$. So $\|f_n - 0\|_\infty = e^{-1}$, which does not converge to 0. [11 points]

(d) (f_n) converges uniformly iff given $\epsilon > 0$ there is an N with $\|f_n - f_m\|_\infty \leq \epsilon$ when $m > n \geq N$. [8 points]

5. (a) $\sum_{k=1}^{\infty} f_k$ converges uniformly on S if the sequence $s_n = \sum_{k=1}^{\infty} f_k$ converges uniformly on S in the sense of 4(a) above (or of the classnotes). [6 points]

(b) The Weierstrass M-test states that if $\|f_n\|_\infty \leq M_n$ and $\sum_n M_n < \infty$ (that is, $\sum_n M_n$ converges), then $\sum_{k=1}^{\infty} f_k$ converges uniformly . Proof: Since $\sum_n M_n$ converges its partial sums are Cauchy so given $\epsilon > 0$ there is an N with

$$\|f_{n+1} + f_{n+1} + \cdots + f_m\|_\infty \leq M_{n+1} + M_{n+1} + \cdots + M_m \leq \epsilon, \qquad m > n \geq N.$$

So $\sum_{k=1}^{\infty} f_k$ satisfies the condition in the Cauchy test for uniform convergence of a series from class, so it is uniformly convergent by that test. [Note this is different from Paulsens notes version which some may give.] [22 points]

(c) We use the the Calculus I technique: if $f_n = \frac{x}{1+n^4x^2}$ then $f_n' = \frac{1+n^4x^2-2n^4x^2}{(1+n^4x^2)^2} = 0$ when $x = \pm\frac{1}{n^2}$. And $|f_n(\pm\frac{1}{n^2})| = \frac{1}{2n^2}$. So $\|f_n'\|_\infty = \frac{1}{2n^2}$; and $\sum_n \frac{1}{2n^2}$ converges by the p-series test. So by the Weierstrass M-test $\sum_{n=1}^{\infty} \frac{x}{1+n^4x^2}$ converges uniformly on $\mathbb{R}$. Since f_n is continuous, the theorem on continuity of a series (Q 4b above), $\sum_k f_k$ is continuous on $\mathbb{R}$. [9 points]

(d) If $f_n' \to g$ uniformly on (a,b) and $(f_n(x_0))$ converges for at least one point $x_0 \in (a,b)$ then f_n converges uniformly to a differentiable function f on (a,b), and $f' = g$ on (a,b). [Also acceptable: if $f_n \to f$ pointwise on (a,b) and $f_n' \to g$ uniformly on (a,b), then f is differentiable function on (a,b), and $f' = g$ on (a,b).] [8 points]

6. (a) The radius of convergence of $\sum_k a_k x^k$ is $1/\limsup_k |a_k|^{\frac{1}{k}}$ where we interpret $1/0$ as ∞ and $1/\infty$ as 0. [6 points]

(b) Here (a_k) is the sequence $0, 3, 0, 3^2, 0, 3^3, \cdots$ so $\limsup_k |a_k|^{\frac{1}{k}} = \sqrt{3}$, so the radius of convergence is $1/\sqrt{3}$. [10 points]

(c) The main theorem about the derivative of a power series $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ with radius of convergence $r > 0$ states that the sum function $f(x)$ is differentiable on $B(x_0, r)$, and its derivative there equals $\sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}$, which is a power series with the same radius of convergence r. Proof in classnotes (sketch: $\limsup_k |ka_k|^{\frac{1}{k}} = \limsup_k |a_k|^{\frac{1}{k}}$ since $k^{\frac{1}{k}} \to 1$ (See Q 3a), so the two series have the same radius of convergence r. So by the theorem about uniform convergence of power series from class, the two series converge uniformly on $B(x_0, s)$ for any positive $s < r$. By the series version of Q 5d from class, $f'(x) = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}$ on $B(x_0, s)$ for all $s < r$, and hence on $B(x_0, r)$. [24 points]

(d) The radius of convergence is $1/(\limsup_k k^{\frac{1}{k}}) = 1$ so the series is differentiable on $(-1, 1)$ with derivative $\sum_{n=1}^{\infty} n^2 x^{n-1}$ here. [10 points]

7. (a) $(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt$. If the support of f is contained in $[a, b]$ then $f(t)$ and $g(x-t)$ are continuous and hence bounded on $[a, b]$, hence $\int_{-\infty}^{\infty} f(t)g(x-t)dt = \int_a^b f(t)g(x-t)dt$ exists. [12 points]

(b) $(f * (g + ch))(x) = \int f(t)(g + ch)(x - t)dt = \int f(t)g(x - t)dt + c \int f(t)h(x - t)dt = (f * g + c(f * h))(x)$. [5 points]

(c) The (real scalar case of the) Stone-Weierstrass theorem: Suppose that K is a compact metric space and $\mathcal{A} \subset C(K)$. Suppose that $\mathcal{A}$ contains constant functions, and $f + g$ and fg are in $\mathcal{A}$ whenever $f, g \in \mathcal{A}$. Suppose also that $\mathcal{A}$ separates points of K. Then $\mathcal{A}$ is dense in $C(K)$. Two other equivalent formulations of the last line are 1) given $f \in C(K)$ and $\epsilon > 0$, there is a $g \in \mathcal{A}$ with $\|f - g\|_\infty < \epsilon$; and 2) given $f \in C(K)$ there is a sequence $g_n \in \mathcal{A}$ which converges uniformly to f. To see that these are equivalent apply in the metric space $C(K)$ with metric $\|f - g\|_\infty$ the principle from 4331 that a set E being dense in a metric space (X, d) (that is, $X = \overline{E}$) is equivalent to (1) every element in X is the limit of a sequence from E, or to (2) given $f \in X$ and $\epsilon > 0$, there is a $g \in E$ with $d(f, g) < \epsilon$ (that is, $B(f, \epsilon) \cap E \neq \emptyset$). [20 points]

(d) Let $\mathcal{A}$ be the polynomials in n variables viewed as functions on K. Then $\mathcal{A}$ satisfies all the conditions of the Stone-Weierstrass theorem: the sum and product of two such polynomials is another such polynomial, and constant functions are polynomials. To see that $\mathcal{A}$ separates points of K: if $\vec{z} \neq \vec{y}$ in K, then for some i we have $z_i \neq y_i$, so the polynomial $\vec{x} \mapsto x_i$ takes different values at these two points. So $\mathcal{A}$ is dense. [14 points]