Department of Mathematics, University of Houston
4332 - Intro to Real Analysis Second Semester - Blecher
Test 2–Spring 2015

Instructions: Put all your bags and papers on the side of the room. Answer question 0, and any one question from questions 1–3, and any one question from questions 4–6. Besides these, do not attempt parts of other questions (they will not be graded). SHOW ALL YOUR REASONING. Time: 85 minutes. In the relevant questions below you may either state the real case or the complex case but you do not need to state both. You may quote freely any results from the notes without proof, except those you are asked to prove. Formulae: Fourier coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\cos(nx)dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\sin(nx)dx, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} fe^{-inx}\,dx, \quad a_0 = c_0.$$

Parseval's identity says $\|f\|_2^2 = \pi(2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2))$ (in the complex case $\|f\|_2^2 = 2\pi\sum_{n=-\infty}^{\infty}|c_n|^2$).

0. (a) Define the Fourier series of a function f on $[-\pi,\pi]$. Also, what is meant by $s_N(f)(x)$? [4]

   (b) Let $h, h_1, h_2, h_3, \cdots$ be a sequence of functions which are Riemann integrable on $[a,b]$, and $a \leq c < d \leq b$. Prove using the Cauchy-Schwarz inequality that if $h_n \to h$ in 2-norm on $[a,b]$ then $\int_c^d h_n\,dx \to \int_c^d h\,dx$. Deduce that if, instead, $\sum_{k=1}^{\infty} h_k = h$ in 2-norm, then $\int_c^d h\,dx = \sum_{k=1}^{\infty}\int_c^d h_n\,dx$. [6+4]

   (c) Let $f(x) = |x|$ for $-\pi \leq x < \pi$, and periodic of period 2π. Show that the Fourier series of f is $\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos((2n-1)x)}{(2n-1)^2}$. In what senses does this Fourier series converge? [10]

   (d) Describe the pointwise limit function of the Fourier series in (c) at every point. Explain why, showing how (with all reasoning) you are applying the results you are using. [6]

   (e) Using (c) and (d) prove that $\frac{\pi}{2} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$. [3]

   (f) Applying Parseval's equation to (c) find a formula for $1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots$. [4]

1. (a) State the periodic Weierstrass theorem. [5]

   (b) What is meant by an orthonormal family of functions on $[a,b]$? [4]

   (c) Write down an orthonormal family of functions on $[-\pi,\pi]$. [3]

   (d) State and prove Bessel's inequality. In your proof state the part of the proof of the 'Theorem on best approximation' that you are using, so that the logic is complete. [6+5]

2. (a) What is meant by the Minkowski inequality? [2]

   (b) Prove that $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx}dx = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$ [3]

   (c) Prove the 'Theorem on best approximation'. [16]

   (d) Under what conditions on a function $f : [-\pi,\pi] \to \mathbb{R}$ does the Fourier series of f converges to f in 2-norm? [1]

3. (a) Define $\|f\|_2$, and prove it is finite and is a norm on $C([a,b])$ (state but you need not prove the triangle inequality).

   (b) Show that if functions $f_n \to f$ uniformly on $[a,b]$ then $f_n \to f$ in 2-norm on $[a,b]$.

   (c) Prove that if f is Riemann integrable on $[-\pi,\pi]$ and if the sum of the Fourier coefficients of f is absolutely convergent, then the Fourier series of f converges uniformly to a continuous function g on $[-\pi,\pi]$, and $\|f-g\|_2 = 0$.

   (d) If f is a differentiable 2π-periodic function with f' Riemann integrable on $[-\pi,\pi]$, then what can one conclude? In particular, what can you say about the convergence of the Fourier series? You do not need to prove anything. [23]

4. (a) What does it mean for a function f to satisfies a Lipschitz continuity condition at x? [4]

   (b) Show that if f is differentiable at a point x then it satisfies a Lipschitz continuity condition at x. (Hint: use the ϵ-δ definition of $\lim_{y\to x} \frac{f(y)-f(x)}{y-x} = f'(x)$, with $\epsilon = 1$.) [8]

   (c) State the result about when a Lipschitz continuity condition of f at x implies something about the Fourier series of f. [5]

   (d) Deduce from (b) and (c) the result from class about when differentiability of f at x implies something about convergence of the Fourier series of f at x. [4]

5. [21]

   (a) State and prove the 'Localization theorem'.

   (b) If s_n is the nth partial sum of a series $\sum_k a_k$ of numbers, then what does it mean for $\sum_k a_k$ to be Cesáro summable? And in this case what is the Cesáro sum?

   (c) If $\sum_k a_k$ converges with sum s then what can you say about the Cesáro sum of $\sum_k a_k$? Also, state a condition that implies that if the Cesáro sum of $\sum_k a_k$ is s then $\sum_k a_k$ converges with sum s.

   (d) The second half of Item (c) is used in the proof of the following result in whose statement you should fill in the blanks: "If f is ________________________________ function, and if n times the nth Fourier coefficients of f (for all integers n) constitute a ________ set, then ____."

6. (a) Define what it means for a Fourier series to be Cesáro summable at x.

   (b) State Fejer's theorem.

   (c) Deduce from Fejer's theorem that if f is a 2π-periodic function which is Riemann integrable on $[-\pi,\pi]$, and if the Fourier series for f converges pointwise at a point x, and if $f(x^-)$ and $f(x^+)$ exist, then the Fourier series for f converges pointwise at x to $\frac{1}{2}(f(x^-)+f(x^+))$.

   (d) Prove that the convolution on $[-\pi,\pi]$ satisfies $f * g = g * f$. [21]