Department of Mathematics, University of Houston
4332 - Intro to Real Analysis Second Semester - Blecher
Test 2–Spring 2015

Instructions: Put all your bags and papers on the side of the room. Answer question 0, and any one question from questions 1–3, nd any one question from questions 4–6. Besides these, do not attempt parts of other questions (they will not be graded). SHOW ALL YOUR REASONING. Time: 85 minutes. In the relevant questions below you may either state the real case or the complex case but you do not need to state both. You may quote freely any results from the notes without proof, except those you are asked to prove. Formulae: Fourier coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\cos(nx)dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\sin(nx)dx, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} fe^{-inx}\,dx, \quad a_0 = c_0.$$

Parseval's identity says $\|f\|_2^2 = \pi(2a_0^2 + \sum_{n=1}^{\infty}(a_n^2+b_n^2))$ (in the complex case $\|f\|_2^2 = 2\pi\sum_{n=-\infty}^{\infty}|c_n|^2$).

0. (a) The Fourier series of f on $[-\pi,\pi]$ is $\sum_{n=-\infty}^{\infty} c_k\,e^{ikx}$, and $s_N(f)(x)$ is its Nth partial sum, namely $\sum_{k=-N}^{N} c_k\,e^{ikx}$. (These are in complex form, most will write the real form.)

(b) For the first part, we have

$$|\int_c^d h_n\,dx - \int_c^d h\,dx| \leq \int_c^d |h_n-h|\,dx \leq \int_a^b 1\cdot|h_n-h|\,dx \leq \|1\|_2\|h_n-h\|_2 \to 0,$$

where we have used the Cauchy-Schwarz inequality for integrals. So $\int_c^d h_n\,dx \to \int_c^d h\,dx$. If instead $\sum_{k=1}^{\infty} h_k = h$ in 2-norm, let $s_n = \sum_{k=1}^{n} h_k$, then $s_n \to h$ in 2-norm. So by the first part applied to s_n we have $\int_c^d s_n\,dx = \sum_{k=1}^{n}\int_c^d h_k\,dx \to \int_c^d h\,dx$. Thus $\sum_{k=1}^{\infty}\int_c^d h_n\,dx = \int_c^d h\,dx$.

(c) This is an even function, and so $f(x)\sin(kx)$ is odd, so $b_k = 0$ for all $k\in\mathbb{N}$. Similarly, $f(x)\cos(kx)$ is even, so if $k\in\mathbb{N}$ then

$$a_k = \frac{2}{\pi}\int_0^{\pi} x\cos(kx)dx = \frac{2}{\pi}(x\sin(kx)/k]_0^{\pi} - \int_0^{\pi}\sin(kx)/kdx = \frac{2}{k^2\pi}\cos(kx)]_0^{\pi} = \frac{2}{\pi}\cdot\frac{(-1)^k-1}{k^2}$$

by Calculus (integration by parts). Clearly $a_0 = \frac{1}{\pi}\int_0^{\pi} xdx = \frac{\pi}{2}$. So the Fourier series of f is $\frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos((2k-1)x)}{(2k-1)^2}$. It converges in the 2-norm, uniformly, pointwise (and Cesaro).

(d) It is $|x|$ at every point. This follows from any one of several of the results from class, eg. Corollary 4, or 4.4, 4.10, or 4.11. In each case you must show how you are using the result. For example if they are using Corollary 4 they must say that f is continuous and the sum of the Fourier coefficients of f is absolutely convergent: $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \leq \sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p-series test and comparison test.

(e) By (d) with $x=0$ we have $0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$, so that $\frac{\pi}{2} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$.

(f) By Parseval's equation in (c) we have

$$\int_{-\pi}^{\pi} x^2dx = 2\int_0^{\pi} x^2dx = \frac{2\pi^3}{3} = \pi(2(\frac{\pi}{2})^2 + (\frac{4}{\pi})^2\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^3}{2} + \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4},$$

so $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^2}{6\cdot 16} = \frac{\pi^2}{96}$. [Points: 4+10+10+6+3+4]

1. (a) If f is a continuous 2π-periodic function, and if $\epsilon > 0$ is given, then there exists a trig polynomial P on $[-\pi, \pi]$ with $|P(x) - f(x)| < \epsilon$ for all $x \in [-\pi, \pi]$.

   (b) An orthonormal family of functions on $[a, b]$ is a set $\{f_1, \cdots f_n\}$ of functions on $[a, b]$ with $\int_a^b f_i \overline{f_j} dx = 0$ if $i \neq j$, and is 1 if $i = j$. (The real case, with no 'bar', is also OK).

   (c) $\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx : n = 1, 2, \cdots\}$ (in the complex case $\{\frac{1}{\sqrt{2\pi}} e^{inx} : n = \cdots, -2, -1, 0, 1, 2, \cdots\}$).

   (d) State and prove Bessel's inequality. In your proof you must state the part of the proof of the 'Theorem on best approximation' that you are using, so that the logic is complete. See Classnotes. [Points: 5+4+3+11]

2. (a) The Minkowski inequality is $\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$. (It is the triangle inequality for the 2-norm.)

   (b) If $k = 0$ we get $\frac{1}{2\pi} \int_{-\pi}^{\pi} 1 dx = 1$. Otherwise $\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} dx = \frac{1}{ik} \frac{1}{2\pi} e^{ikx}]_{x=-\pi}^{\pi} = 0$. (Or one can do this like $\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(kx) dx + i \int_{-\pi}^{\pi} \sin(kx) dx = \cdots$.)

   (c) Prove the 'Theorem on best approximation'. See class notes.

   (d) Riemann integrable. [Points: 2+3+16+1]

3. (a) $\|f\|_2 = \sqrt{\int_a^b |f|^2 dx} \geq 0$, and if $= 0$ iff $|f|^2 = 0$ by Math 3333, so $f = 0$. Also $|f|$ is bounded by a constant M since $[a, b]$ is compact by 3333/4331 so $\int_a^b |f|^2 dx \leq M(b - a) < \infty$ (or one can use f Riemann integrable implies f^2 integrable by 3333/4331). Then $\|cf\|_2 = \sqrt{\int_a^b |cf|^2 dx} = |c| \sqrt{\int_a^b |f|^2 dx}$ for a scalar c. The triangle inequality says $\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$.

   (b) This follows because $\int_a^b |f_n - f|^2 dx \leq \int_a^b \|f_n - f\|_\infty^2 dx = (b - a)\|f_n - f\|_\infty^2$. So if $f_n \to f$ uniformly on $[a, b]$ then $\|f_n - f\|_2 \leq \sqrt{b - a}\ \|f_n - f\|_\infty \to 0$ as $n \to \infty$, so by 'squeezing' $\|f_n - f\|_2 \to 0$.

   (c) We just do the complex case, the real case is almost identical. Since $|c_k e^{ikx}| \leq |c_k|$, if $\sum_{k=-\infty}^{\infty} |c_k| < \infty$ then by the Weierstrass M-test the Fourier series $\sum_{k=-\infty}^{\infty} c_k\, e^{ikx}$ of f converges uniformly to a function g on $[-\pi, \pi]$, and by the theorem on continuity of infinite sum functions, g is continuous. So $s_N(f) \to g$ in 2-norm by (b). Then

$$\|f - g\|_2 \leq \|f - s_N(f)\|_2 + \|s_N(f) - g\|_2 \to 0,$$

   so that $\|f - g\|_2 = 0$.

   (d) If f is a differentiable 2π-periodic function with f' Riemann integrable on $[-\pi, \pi]$, then the sum of the Fourier coefficients of f is absolutely convergent, and the Fourier series of f converges uniformly to f on $[-\pi, \pi]$. [23]

4. (a) There exists positive numbers $M, \delta > 0$ such that $|f(y) - f(x)| \leq M|y - x|$ whenever $|y - x| < \delta$.

(b) Using the ϵ-δ definition of $\lim_{y\to x} \frac{f(y)-f(x)}{y-x} = f'(x)$, with $\epsilon = 1$, there is a $\delta > 0$ such that

$$|\frac{f(y)-f(x)}{y-x} - f'(x)| = |\frac{f(y)-f(x)-f'(x)(y-x)}{y-x}| < 1, \qquad |y-x| < \delta.$$

Multiplying by $|y-x|$ and using the triangle inequality shows that if $|y-x| < \delta$,

$$|f(y)-f(x)| \le |f(y)-f(x)-f'(x)(y-x)| + |f'(x)(y-x)| < |y-x| + |f'(x)(y-x)| = M|y-x|,$$

where $M = 1 + |f'(x)|$.)

(c) Suppose that f is a 2π-periodic function which is Riemann integrable on $[-\pi,\pi]$ and satisfies a Lipschitz continuity condition at a number x. Then the Fourier series of f converges to $f(x)$ at x.

(d) By (b) if f is differentiable at x the Lipschitz condition at x in (a) holds, so by (c) the Fourier series of f converges to $f(x)$ at x (if f is a 2π-periodic function which is Riemann integrable on $[-\pi,\pi]$). [Points: 4+8+5+4]

5. (a) The 'Localization theorem' states that if f and g are 2π-periodic functions which are Riemann integrable on $[-\pi,\pi]$, and if $f = g$ on an open interval J and , if $x \in J$ then the Fourier series of f and g at x either both converge to the same value, or both diverge. Proof: Let $h = f - g$, then $h = 0$ on J. So by Corollary 4.5 the Fourier series of h converges to 0 at any $x \in J$. But the Fourier series of h is the Fourier series of f minus the Fourier series of g. It is clear from this.

(b) If s_n is the nth partial sum of a series $\sum_k a_k$ of numbers, then $\sum_k a_k$ is Cesáro summable if the sequence (σ_n) converges, where $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. In this case the Cesáro sum is $\lim_n \sigma_n$.

(c) If $\sum_k a_k$ converges with sum s then the Cesáro sum of $\sum_k a_k$ equals s. A homework 11 question states that if the Cesáro sum of $\sum_k a_k$ is s, and $(n(s_n - s_{n-1}))$ is bounded, then $\sum_k a_k$ converges with sum s.

(d) "If f is [Riemann integrable on $[-\pi,\pi]$ and is a 2π-periodic function], and if n times the nth Fourier coefficients of f (for all integers n) constitute a BOUNDED set, then [the Fourier series for f converges pointwise at x to $\frac{1}{2}(f(x^-) + f(x^+))$ for every x for which $f(x^-)$ and $f(x^+)$ exist.] [Points: 21]

6. (a) A Fourier series to be Cesáro summable at x if the sequence $(\sigma_N(f)(x))$ converges as $N \to \infty$, where $\sigma_N(f)(x) = \frac{s_1(f)(x)+s_2(f)(x)+\cdots+s_N(f)(x)}{N}$.

(b) Fejer's theorem states that if f is a 2π-periodic function which is Riemann integrable on $[-\pi,\pi]$, then at any point x where $f(x^-)$ and $f(x^+)$ exist, the Fourier series of f is Cesáro summable at x, and its Cesáro sum at x is $\lim_{n\to\infty} \sigma_n(f)(x) = \frac{1}{2}(f(x^-) + f(x^+))$.

(c) This follows from Fejer's theorem (b) above together with the fact mentioned before Fejer's theorem in the notes (related to 5(c) above) that if the Fourier series converges pointwise at x with sum s, then it is Cesáro summable at x and its Cesáro sum at x is s. the Fourier series for f converges pointwise at x to $\frac{1}{2}(f(x^-) + f(x^+))$.

(d) The convolution on $[-\pi, \pi]$ is $(f * g)(x) = \int_{-\pi}^{\pi} f(y)g(x-y)dy$. Let $u = x - y$ for fixed x, then $du = -dy$, and $y = x - u$, and the integral becomes

$$-\int_{x+\pi}^{x-\pi} f(x-u)g(u)dy = \int_{x-\pi}^{x+\pi} g(u)f(x-u)dy = \int_{-\pi}^{\pi} g(u)f(x-u)dy = (g * f)(x).$$

In the last integral we have used the fact that $g(u)f(x-u)$ is 2π-periodic, and for any c-periodic function h and any real numbers d and b, we have $\int_d^{d+c} hdt = \int_b^{b+c} hdt$. [Points: 21]