Department of Mathematics, University of Houston
4332 - Intro to Real Analysis Second Semester - Blecher
Mock Exam for Test 1

Instructions: Put all your bags and papers on the side of the room. Answer question 0 and any two other questions. SHOW ALL YOUR REASONING. You may quote freely any results from the notes without proof, except those you are asked to prove.

0. (a) Suppose that $\sum_{k=1}^{\infty} a_k$ converges. Does $a_1 + a_2 + \cdots + a_{10} - a_1 + a_{11} + a_{12} + \cdots + a_{20} - a_2 + a_{21} + \cdots$ converge? If so, prove it, and find the sum; if not find a counterexample.
   (b) Show that $g_n(x) = x^n(1 - x^2)$ converges uniformly on $[0, 1]$.
   (c) Find the interval of convergence of $\sum_{k=1}^{\infty} 2^k x^{k^3}$.
   (d) Let $f(x) = \sum_{k=1}^{\infty} \frac{\sin(2^k x)}{3^k}$, for all x where this series converges. Find where f is continuous, where it is differentiable, and calculate $f'(0)$. (SHOW ALL REASONING).
   (e) State the Weierstrass polynomial approximation theorem, and give two other equivalent formulations of it. Then show how it follows from the Stone-Weierstrass theorem.

1. (a) What does it mean for a series of real numbers to be convergent? Absolutely convergent?
   (b) State the Cauchy criterion for convergence of a series of real numbers.
   (c) Prove that an absolutely convergent series of real numbers converges and then $|\sum_{k=1}^{\infty} a_k| \leq \sum_{k=1}^{\infty} |a_k|$.
   (d) Show that $\sum_{k=2}^{\infty} \frac{(-1)^k}{k(\log k)^2}$ converges, and converges absolutely.

2. (a) Define an infinite *double series* of real numbers $\sum_{n,m=1}^{\infty} a_{nm}$, and say what it means for such to converge.
   (b) Prove that if $\sum_{n,m=1}^{\infty} a_{nm}$ and $\sum_{n,m=1}^{\infty} b_{nm}$ converge, then so does $\sum_{n,m=1}^{\infty}(a_{nm} + b_{nm})$.
   (c) Prove that if both $\sum_{n=1}^{\infty} a_n$ and $\sum_{m=1}^{\infty} b_m$ converge, then so does $\sum_{n,m=1}^{\infty} a_n b_m$.
   (d) Using the Cauchy product, prove that $e^{x+y} = e^x e^y$, if e^x is defined to be $\sum_{k=0}^{\infty} \frac{x^n}{n!}$.

3. (a) Define *uniform convergence of a sequence of real valued functions* on a subset S of $\mathbb{R}$.
   (b) Show that if $f_n \to f$ uniformly on S , and if each f_n is uniformly continuous on S , then f is uniformly continuous on S .
   (c) Show that $(\frac{x}{1+nx^2})$ converges uniformly on $\mathbb{R}$.
   (d) State Dini's theorem.

4. (a) Define what it means for a sequence $\{f_n\}$ of real valued functions defined on a set Ω to be *uniformly Cauchy*.

(b) Prove that a uniformly Cauchy sequence of functions is uniformly convergent.

(c) Prove that the set $B(\Omega)$ of bounded real-valued functions on a set Ω is a complete metric space with metric $d(f,g) = \sup\{|f(\omega) - g(\omega)| : \omega \in \Omega\}$.

(d) State some conditions which will ensure that $\sum_{k=1}^{\infty}(\int_a^b f_n(x)dx) = \int_a^b(\sum_{k=1}^{\infty} f_n(x))dx$.

5. (a) Define *uniform convergence of a series of real valued functions* on a subset S of $\mathbb{R}$.

(b) State and prove the Weierstrass M-test.

(c) Show that $\sum_{n=0}^{\infty} \frac{x^n}{1+n^2x^2}$ is continuous on [-1,1].

(d) Give conditions ensuring that if $f_n \to f$ then f is differentiable and $\lim_{n\to\infty} f_n' = f'$.

6. Construct a curve $\gamma : [0,1] \to [0,1] \times [0,1]$ which is continuous and onto. Prove that any such curve cannot be 1-1.

7. (a) Define the radius of convergence of a power series.

(b) Show that a power series $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ with positive radius of convergence r is differentiable on $(x_0 - r, x_0 + r)$, with $f'(x) = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}$ here. Also show that if K is a compact subset of $(x_0 - r, x_0 + r)$, then this last series converges uniformly on K.

(c) Find where the series $\sum_{n=1}^{\infty}(x/n)^n$ is differentiable, and give a formula for the derivative here.