Department of Mathematics, University of Houston
4332 - Intro to Real Analysis Second Semester - Blecher
Incomplete Mock Exam for Test 2

Instructions: Put all your bags and papers on the side of the room. Answer question 0 and any two other questions. Besides these, do not attempt parts of other questions (they will not be graded). You may quote freely any results from the notes without proof, except those you are asked to prove. SHOW ALL YOUR REASONING. Time: 85 minutes.

0. (a) State Parseval's equation (not the Corollary labelled in the notes as 'Parseval'). Under what conditions on f does it hold?
   (b) Give a short survey of the conditions under which the Fourier series of a function will converge (in various senses) to the function.
   (c) Show that $x = \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ if $0 < x < 2\pi$.
   (d) Using Parseval's equation in (c) find $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
   (e) Using (c) and (d) show that $\frac{x^2}{2} = \pi x - \frac{\pi^2}{3} + 2\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ if $0 \leq x \leq 2\pi$.
   (f) Using (e) find $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

1. (a) Define $||.||_2$ for functions on [a,b].
   (b) State the Cauchy-Schwarz inequality for integrals of functions on [a,b].
   (c) Prove that $||.||_2$ is a seminorm on the Riemann integrable functions on [a,b], and is a norm on the continuous functions on [a,b]. In particular, prove the triangle inequality for $||.||_2$. What is another name for this triangle inequality?
   (d) Prove that if we have a sequence of Riemann integrable functions $h, h_1, h_2, h_3, \cdots$ which are Riemann integrable on $[a, b]$, and if $h_n \to h$ in 2-norm on $[a, b]$, and $a \leq c < d \leq b$, then $\int_c^d h_n\, dx \to \int_c^d h\, dx$. [Hint: use the Cauchy-Schwarz inequality for integrals.] Deduce that if instead of $h_n \to h$ in 2-norm, we have that $\sum_{k=1}^{\infty} h_k = h$ in 2-norm, then $\int_c^d h\, dx = \sum_{k=1}^{\infty} \int_c^d h_n\, dx$.

2. (a) What is a trigonometric polynomial of degree $\leq N$?
   (b) State and prove the 'orthogonality relations'.
   (c) Prove that if $\alpha_0 + \sum_{k=1}^{n} (\alpha_k cos(kx) + \beta_k \sin(kx)) = 0$ for all $x \in [0, 2\pi]$, and α_k and β_k are constants and $n \in \mathbb{N}$, then all the α_k and β_k are zero.
   (d) Prove that if f is Riemann integrable on $[0, 2\pi]$, and if $\alpha_0 + \sum_{k=1}^{\infty} (\alpha_k cos(kx) + \beta_k \sin(kx)) = f(x)$ in 2-norm on $[0, 2\pi]$, where α_k and β_k are constants, then α_k and β_k are the Fourier coefficients of f.

3. (a) What does it mean for a function f to satisfies a Lipschitz continuity condition at x?

(b) Show that if a function is differentiable at a point x then it satisfies a Lipschitz continuity condition at x. (Hint: look at the ϵ-δ definition of $\lim_{y\to x} \frac{f(y)-f(x)}{y-x} = f'(x)$, with $\epsilon = 1$.)

(c) State the result about when a Lipschitz continuity condition of f at x implies something about the Fourier series of f.

(d) Deduce from (b) and (d) the result about when differentiability at x implies something about the Fourier series of f.

4. (This is just an additional practice problem.) Let $0 < \delta < \pi$, $f(x) = 1$ if $|x| \le \delta$, $f(x) = 0$ if $\delta < |x| \le \pi$, and f is 2π-periodic.

   (a) Find the Fourier coefficients of f

   (b) Prove $\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi-\delta}{2}$ if $0 < \delta < \pi$.

   (c) Using Parseval show that $\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi-\delta}{2}$.

   (d) Let $\delta \to 0$ and deduce that $\int_0^{\infty} (\frac{\sin x}{x})^2 \, dx = \frac{\pi}{2}$. (This one is harder, and optional.)

   (e) Put $\delta = \frac{\pi}{2}$ in (c) above. What do you get?

5. Another 4 part question of a similar type to Test 1 (state definitions, state a theorem or result from the classnotes, questions taken more or less directly from the turned in Homework, or prove a theorem or result from the classnotes from the list of proofs on the website.

6. Another 4 part question of a similar type to Test 1 (state definitions, state a theorem or result from the classnotes, questions taken more or less directly from the turned in Homework, or prove a theorem or result from the classnotes from the list of proofs on the website.