Department of Mathematics, University of Houston
4332 - Intro to Real Analysis Second Semester - Blecher
Incomplete Mock Exam for Test 3

Instructions: Put all your bags and papers on the side of the room. Answer question 0 and any two other questions. Besides these, do not attempt parts of other questions (they will not be graded). You may quote freely any results from the notes without proof, except those you are asked to prove. SHOW ALL YOUR REASONING. Time: 50 minutes.

0. (a) Find $f'(0,0)$ if it exists if $f(x,y) = \sqrt{|xy|}$.
   (b) Let $f(u,v,w) = (e^{u-w}, \cos(v+u) + \sin(u+v+w))$ and $g(x,y) = (e^x, \cos(y-x), e^{-y})$. Calculate $(f \circ g)'(0,0)$ using our chain rule.
   (c) State the inverse function theorem and prove it's first assertion.
   (d) Consider the system

$$\begin{cases} u = x + xyz \\ v = y + xy \\ w = z + 2x + 3z^2. \end{cases}$$

   (i) Show using the inverse function theorem that it is possible to solve for x, y, and z above explicitly in terms of u, v, w, in a neighborhood of the point $(x,y,z) = (0,0,0)$.
   (ii) Using the inverse function theorem find the Jacobian matrix of derivatives of x, y, and z with respect to u, v, w, at the point $(0,0,0)$.
   (iii) Find the first partial derivatives of x with respect to u, v, w, at the point where $(x,y,z) = (0,0,0)$.

1. (a) Define the total derivative.
   (b) Prove that a function has at most one total derivative matrix. That is, show that if f is differentiable at a point then the derivative matrix is unique.
   (c) Indeed prove that if f is differentiable at a point then the derivative matrix is its Jacobian matrix.
   (d) Let $f(x,y) = \frac{x^4}{x^2+y^2}$ except at the origin, where $f(0,0) = 0$. Is f differentiable at $(0,0)$? Prove it.

2. (a) State and prove a mean value theorem for a differentiable function $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ on an open set U.
   (b) Show that if f in (a) has $f'(\vec{x}) = O$ for all $\vec{x} \in U$, and if U is path connected, then f is constant.
   (c) Define the norm of a matrix.

(d) Show that if f in (a) has $\|f'(\vec{x})\| \leq M$ for all $\vec{x} \in U$, then $\|f(\vec{x}) - f(\vec{y})\| \leq M\|\vec{x} - \vec{y}\|$ for all $\vec{x}, \vec{y} \in U$. You may assume U is convex.

3. (a) What does it mean for a function $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ to be class C^1 on the open set U?
   (b) State the implicit function theorem.
   (c) Prove that there are functions $u(x, y), v(x, y)$, and $w(x, y)$, and an $r > 0$ such that u, v, w are C^1 on $B((1, 1), r)$, satisfy $u(1, 1) = 1, v(1, 1) = 1, w(1, 1) = -1$, and satisfy the equations
$$\begin{cases} u^5 + xv^2 - y + w = 0 \\ v^5 + yu^2 - x + w = 0 \\ w^4 + y^5 - x^4 = 1. \end{cases}$$
   (d) Use the implicit function theorem to compute $\frac{\partial v}{\partial y}$.

4. Another 4 part question of a similar type to Test 1 or Test 2 (state definitions, state a theorem or result from the classnotes, questions taken more or less directly from the turned in Homework, or prove a theorem or result from the classnotes from the list of proofs on the website.

5. Another 4 part question of a similar type to Test 1 or Test 2 (state definitions, state a theorem or result from the classnotes, questions taken more or less directly from the turned in Homework, or prove a theorem or result from the classnotes from the list of proofs on the website.