INTRODUCTION TO REAL ANALYSIS II
MATH 4332–BLECHER NOTES

1. As in earlier classnotes

2. As in earlier classnotes (Fourier series)

3. Fourier series (continued)

(NOTE: UNDERGRADS IN THE CLASS ARE NOT RESPONSIBLE FOR KNOWING LARGE SECTIONS OF THE FOLLOWING NOTES, AS EXPLAINED ON PAGE 3 BELOW.)

... As in earlier classnotes ...

Last time we proved:

Theorem If f is Riemann integrable on $[-\pi, \pi]$ then the Fourier series of f converges to f in 2-norm. That is, $\|s_N(f) - f\|_2 \to 0$ as $N \to \infty$, where $s_N(f)$ as usual is the Nth partial sum of the Fourier series. Also,

$$\|f\|_2^2 = \pi(2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2))$$

(in complex case $\|f\|_2^2 = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2$). Here a_n, b_n, c_n are the Fourier coefficients for f.

This formula is also called *Parseval's identity.*

Corollary 1 If f and g are two continuous scalar valued functions on $[-\pi, \pi]$ with the same Fourier series (or equivalently, each Fourier coefficient for f is the same as the matching Fourier coefficient for g), then $f = g$ on $[-\pi, \pi]$.

Proof. Let $h = f - g$. Every Fourier coefficient for h is the difference between the-Fourier coefficient for f and the matching Fourier coefficient for g, by the linearity of the integral in the formula for the Fourier coefficient, and so is 0. For example, the nth complex Fourier coefficient of h is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} he^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f-g)\,e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\,e^{-inx}\,dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} g\,e^{-inx}\,dx.$$

This is the difference between the nth Fourier coefficient for f and the nth Fourier coefficient for g, which is 0. By the formula above this Corollary, the Parseval identity, we get that $\|h\|_2^2 = \int |h(x)|^2 \, dx = 0$. Since h is continuous, by 3333 we get that $|h(x)|^2 = 0$ at every x, so that $f(x) = g(x)$ at every x. □

Corollary 2 (The Riemann-Lebesgue lemma) If f is Riemann integrable on $[-\pi, \pi]$ then the nth Fourier coefficients for f converge to zero as $n \to \infty$.

Proof. By the formula above Corollary 1, or (better) by Bessels inequality, the sum of the squares of the Fourier coefficients converge (in the complex case $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$). So by the Divergence Test in Section 2 of Chapter 0, the Fourier coefficients converge to zero. □

Corollary 3 If f is Riemann integrable on $[-\pi, \pi]$ and the sum of the Fourier coefficients of f is absolutely convergent, then the Fourier series of f converges uniformly to a continuous function g on $[-\pi, \pi]$, and $\|f - g\|_2 = 0$.

Proof. By the proof of Homework 10 Question 3, it is only necessary to do the complex case (the real Fourier series equals the complex Fourier series). Since $|c_k e^{ikx}| \leq |c_k|$, if $\sum_{k=-\infty}^{\infty} |c_k| < \infty$ then by the Weierstrass M-test the Fourier series $\sum_{k=-\infty}^{\infty} c_k \, e^{ikx}$ of f converges uniformly to a function g on $[-\pi, \pi]$, and by the theorem on continuity of infinite sum functions, g is continuous. (Strictly speaking, these results were phrased for series $\sum_{k=1}^{\infty}$ rather than $\sum_{k=-\infty}^{\infty}$, but the latter can easily be rewritten as the former). So $s_N(f) \to g$ in 2-norm by Homework 10 Question 5. Then by the triangle inequality and the Theorem before these Corollaries, we have

$$\|f - g\|_2 = \|f - s_N(f) + s_N(f) - g\|_2 \leq \|f - s_N(f)\|_2 + \|s_N(f) - g\|_2 \to 0,$$

so that $\|f - g\|_2 = 0$. □

Negligible sets: One measure of when a set E of numbers is 'small' or 'negligible' is the concept of *Lebesgue measure zero* . This means that given any $\epsilon > 0$ there exist intervals $I_1, I_2, \cdots$ whose union contains E but the sum of the lengths of these intervals is smaller than ϵ. Lebesgue proved that a bounded scalar valued function h on $[a, b]$ is Riemann integrable iff it is continuous except on a set of Lebesgue measure zero. One can also show that such h satisfies $\|h\|_2 = 0$ iff $h = 0$ on $[a, b]$ except on a set of Lebesgue measure zero. In this language, Corollary 3 says that if f is Riemann integrable on $[-\pi, \pi]$ and the sum of the Fourier coefficients of f is absolutely convergent, then the Fourier series of f converges uniformly to a continuous function g on $[-\pi, \pi]$, and g is the same as f except on a set of Lebesgue measure zero.

Corollary 4 If f is continuous and 2π-periodic and the sum of the Fourier coefficients of f is absolutely convergent, then the Fourier series of f converges uniformly to f on $[-\pi, \pi]$.

Proof. By the last result the Fourier series of f converges uniformly to a continuous function g on $[-\pi, \pi]$ and $\|f - g\|_2 = 0$. As we have said before (see e.g. at the end of the proof of the Corollary 1 above), this implies that $f = g$. So the Fourier series of f converges uniformly to f. □

Corollary 5 If f is a differentiable 2π-periodic function with f' Riemann integrable on $[-\pi, \pi]$, then the sum of the Fourier coefficients of f is absolutely convergent, and the Fourier series of f converges uniformly to f on $[-\pi, \pi]$.

Proof. By the proof of Homework 10 Question 3, it is only necessary to do the complex case (the real Fourier series equals the complex Fourier series). Suppose that $\sum_{n=-\infty}^{\infty} d_n \, e^{inx}$ is the Fourier series for f' on $[-\pi, \pi]$. Then

$$d_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(t) \, e^{-int} dt = \frac{1}{2\pi} f(t) \, e^{-int}]_{-\pi}^{\pi} + \frac{in}{2\pi} \int_{-\pi}^{\pi} f \, e^{-inx} dx = inc_n,$$

using integration by parts, since $\frac{d}{dx}(e^{-int}) = -ine^{-int}$. By Parseval or (better) Bessel's applied to f', $\sum_{n=-\infty}^{\infty} |inc_n|^2 = \sum_{n=-\infty}^{\infty} n^2 |c_n|^2 < \infty$ (note $|i|^2 = 1$). We recall the Cauchy Schwartz inequality for scalars

$$|\sum_{k=1}^{n} z_k \, w_k| \leq \sqrt{\sum_{k=1}^{n} |z_k|^2} \sqrt{\sum_{k=1}^{n} |w_k|^2}.$$

One may also take $n = \infty$ in this formula if the right side is then finite. It follows that $\sum_{n=1}^{\infty} |c_n| = \sum_{n=1}^{\infty} \frac{1}{|n|} |n \, c_n| \leq \sqrt{\sum_{n=1}^{\infty} \frac{1}{|n|^2}} \, \sqrt{\sum_{n=1}^{\infty} |n|^2 |c_n|^2} < \infty$, by the Cauchy Schwartz inequality. Similarly, $\sum_{n=-\infty}^{-1} |c_n| < \infty$. So $\sum_{n=-\infty}^{\infty} |c_n| < \infty$.

Note f is continuous since it is differentiable. So by the last paragraph and Corollary 4, the complex Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges uniformly to $f(x)$. □

4. Pointwise convergence of Fourier series

We remark that there are continuous scalar valued functions on $[-\pi, \pi]$ whose Fourier series diverges at infinitely many points. An incredibly deep result of Carleson implies that if f is a Riemann integrable function on $[-\pi, \pi]$ then the Fourier series converges to $f(x)$ for all x except for a set of points of Lebesgue measure zero. We will not say anything about the proof of this result!!

In the results in this section, undergraduates are not responsible for any proofs longer than 5 typed lines below, or for knowing the statements of Lemma's (however those undergraduates planning to go to grad school or who might need Fourier analysis or signals processing are encouraged to read these carefully and understand everything).

Below, we will work with complex numbers throughout, so be sure that you know the basic rules for multiplying and dividing complex numbers $z = a + ib$ and $w = c + id$, etc, like

$$zw = ac - bd + i(ad + bc), \quad |w|^2 = w\bar{w} = c^2 + d^2, \quad \frac{z}{w} = \frac{(a+ib)\overline{w}}{w\bar{w}} = \frac{(ac+bd) + i(bc-ad)}{c^2+d^2}.$$

Here a, b, c, d are real. Recall also that the real part of $z = a + ib$ is $\text{Re}(z) = a$, and the imaginary part is b, and clearly from the formula in the center above we have $|Re(z)| = |a| \leq \sqrt{a^2 + b^2} = |z|$. From the formula $|w|^2 = w\bar{w}$ above it is easy to see that $|wz| = |w||z|$. We proved earlier that $\overline{e^{ix}} = e^{-ix}$ for real numbers x, and of course $|e^{ix}| = 1$ (because the unit circle has radius 1). The symbol $s_N(f)(x)$ will denote the partial sum of the Fourier series of f, so $s_N(f)(x) = \sum_{k=-N}^{N} c_k \, e^{ikx}$, where $c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, e^{-ikt} \, dt$. By the proof of Homework 10 Question 3, it is only necessary to do the complex case (the real Fourier series equals the complex Fourier series, and the 'real' $s_N(f)(x)$ equals the 'complex' $s_N(f)(x)$).

We define the *Dirichlet kernel* to be

$$D_N(x) = \sum_{k=-N}^{N} e^{ikx}.$$

Note that $D_0 = 1$.

Recall that we discussed the convolution earlier, and in Homework 9 Question 4. Below we will use convolution on $[-\pi, \pi]$, namely $(f * g)(x) = \int_{-\pi}^{\pi} f(t) \, g(x-t) \, dt$, for 2π-periodic functions f and g. This has the same properties as the convolution in Homework 9 Question 4, by the same proofs, and also $f * g = g * f$ as before, by essentially the same proof as before (see Notes February 26, and Homework 10 question 10).

Lemma 4.1. (1) $D_N(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}$ *if x is not of form $2m\pi$ for an integer m, and otherwise* $D_N(x) = 2N + 1$.

(2) $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x) \, dx = 1$.

(3) *If f is a Riemann integrable function on $[-\pi, \pi]$ then*

$$s_N(f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, D_N(x - t) \, dt.$$

That is $s_N(f)$ is the convolution $f * (\frac{1}{2\pi} D_N)$.

Proof. (1) Note $\sum_{k=-N}^{N} e^{ikx}$ is a geometric series:

$$D_N(x) = e^{-iNx} + e^{-iNx} \, r + e^{-iNx} \, r^2 + \cdots + e^{-iNx} \, r^{2N}$$

where $r = e^{ix}$. If $r = 1$, that is if x is of form $2m\pi$ for an integer m, then $D_N(x) = 2N + 1$. If $r \neq 1$, that is if x is not of form $2m\pi$ for an integer m, then the sum of this geometric series, by the geometric series formula, is

$$D_N(x) = e^{-iNx}\,\frac{1-r^{2N+1}}{1-r} = r^{-N}\,\frac{1-r^{2N+1}}{1-r} = \frac{r^{N+1}-r^{-N}}{r-1} = \frac{e^{i(N+1)x}-e^{-iNx}}{e^{ix}-1},$$

so that

$$(e^{ix}-1)D_N(x) = e^{i(N+1)x} - e^{-iNx}.$$

Multiply both sides by $e^{-ix/2}$:

$$(e^{ix/2}-e^{-ix/2})\,D_N(x) = e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}.$$

Divide both sides by 2, and use a formula for sin from earlier, to get $\sin(\frac{x}{2})\,D_N(x) = \sin((N+\frac{1}{2})x)$. Thus we see that $D_N(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}$, if $x \neq 2m\pi$.

(2) $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\sum_{k=-N}^{N} e^{ikx})dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1dx + 0 = 1$ using Homework 10 Question 2 (which gives that only the $k=0$ term in the sum has a nonzero integral).

(3) $s_N(f)(x) = \sum_{k=-N}^{N}(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,e^{-ikt}\,dt)\,e^{ikx} = \frac{1}{2\pi}\sum_{k=-N}^{N}(\int_{-\pi}^{\pi} f(t)\,e^{-ikt}e^{ikx}\,dt)$. Moving the sigma symbol inside the integral we see that

$$s_N(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)(\sum_{k=-N}^{N} e^{ik(x-t)})dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,D_N(x-t)\,dt.$$

So $s_N(f) = f * (\frac{1}{2\pi}D_N)$. □

We define the *Fejer kernel* to be

$$K_N(x) = \frac{1}{N+1}\sum_{k=0}^{N} D_k(x), \qquad N = 1, 2, \cdots.$$

Lemma 4.2. (1) $K_N(x) = \frac{1}{N+1}\left(\frac{1-\cos((N+1)x)}{1-\cos(x))}\right)$ *if* $0 < |x| \leq \pi$.

(2) $K_N(x) \geq 0$, *for all* x.

(3) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} K_N(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} K_N(x)\,dx = 1$.

(4) $K_N(x) \leq \frac{2}{(N+1)(1-\cos\delta)}$ *if* $0 < \delta \leq |x| \leq \pi$.

Proof. (1) By an earlier formula for $D_k(x)$, we have

$$K_N(x) = \frac{1}{N+1}\sum_{k=0}^{N}\frac{e^{i(k+1)x}-e^{-ikx}}{e^{ix}-1}.$$

Multiplying both sides by $(N+1)(e^{ix}-1)\,(e^{-ix}-1)$ we get

$$(N+1)K_N(x)(e^{ix}-1)\,(e^{-ix}-1) = (e^{-ix}-1)\sum_{k=0}^{N}(e^{i(k+1)x}-e^{-ikx}).$$

Carefully multiplying out the last expression, we get some kind of telescoping sum which collapses to $2 - e^{i(N+1)x} - e^{-i(N+1)x} = 2(1-\cos((N+1)x)$. Also,

$$(e^{ix}-1)\,(e^{-ix}-1) = 2 - e^{ix} - e^{-ix} = 2(1-\cos x).$$

Thus $K_N(x) = \frac{1}{N+1}\left(\frac{1-\cos((N+1)x)}{1-\cos(x))}\right)$.

(2) This is obvious from (1), and the fact that $D_k(0) = 2k+1 \geq 0$ by Lemma 4.1 (1).

(3) We have using Lemma 4.1 (2) that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\,dx = \frac{1}{N+1}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}(\sum_{k=0}^{N} D_k(x))\,dx = \frac{1}{N+1}\sum_{k=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} D_k(x)\,dx = \frac{N+1}{N+1} = 1.$$

By (1) it is clear that K_N is an even function, and by Calculus 1 for any even function $g(x)$ we have $\int_{-\pi}^{0} g(x)\,dx = \int_0^{\pi} g(x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} g(x)\,dx$.

(4) If $0 < \delta \leq |x| \leq \pi$ then $K_N(x) = \frac{1-\cos((N+1)x)}{(N+1)(1-\cos(x))} \leq \frac{2}{(N+1)(1-\cos(\delta))}$. □

We will use the last results first to give a direct proof of the periodic Weierstrass approximation theorem.

Theorem 4.3. (The periodic Weierstrass approximation theorem) *If f is a continuous 2π-periodic function, and if $\epsilon > 0$ is given, then there exists a trig polynomial P on $[-\pi,\pi]$ with $|P(x) - f(x)| < \epsilon$ for all $x \in [-\pi,\pi]$.*

Proof. Suppose that f is a continuous 2π-periodic function. We copy very closely the direct proof of the polynomial Weierstrass approximation theorem in the class notes for February 26, so follow along with those notes. Let $\epsilon > 0$ be given. Choose an upper bound M for $|f|$, and choose δ as in those class notes such that $|f(x) - f(y)| < \epsilon/3$ whenever $|x-y| < \delta$. Note that the Dirichlet kernels are (complex) trig polynomials by definition, and hence $K_N(x) = \frac{1}{N+1}\sum_{k=0}^{N} D_k(x)$ is also a (complex) trig polynomial. By Lemma 4.1 (3) we have $f * D_k$ is a (complex) trig polynomial (equal to $2\pi\, s_k(f)$). Hence $f * K_N = f * (\frac{1}{N+1}\sum_{k=0}^{N} D_k) = \frac{1}{N+1}\sum_{k=0}^{N} f * D_k$ is also a (complex) trig polynomial. Let $P_N = \frac{1}{2\pi} f * K_N$. It is enough to show that given $\epsilon > 0$ there exists an N with $|P_N(x) - f(x)| < \epsilon$ for all $x \in [-\pi,\pi]$. Now $P_N(x) - f(x) = \frac{1}{2\pi}(K_N * f)(x) - f(x)$, so (following the proof in the class notes at the end of February 26),

$$|P_N(x) - f(x)| = \frac{1}{2\pi}|\int_{-\pi}^{\pi} K_N(t)\,f(x-t)\,dt - \int_{-\pi}^{\pi} K_N(t)\,f(x)dt| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)\,|f(x-t) - f(x)|dt.$$

As in the proof in the class notes at the end of February 26 we split the last integral into $\int_{-\pi}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\pi}$. In the first and third of these integrals $K_N(t) \leq \frac{2}{(N+1)(1-\cos\delta)}$ by Lemma 4.2 (4), and so each of these two integrals is dominated by $\frac{1}{2\pi}(\pi - \delta)\frac{2}{(N+1)(1-\cos\delta)} \cdot 2M$, which for large enough N is smaller than $\frac{\epsilon}{3}$. The middle of the three integrals above is

$$\frac{1}{2\pi}\int_{-\delta}^{\delta} K_N(t)\,|f(x-t) - f(x)|dt \leq \frac{1}{2\pi}\int_{-\delta}^{\delta} K_N(t)\,\frac{\epsilon}{3}\,dt \leq \frac{1}{2\pi}\,\frac{\epsilon}{3}\int_{-\pi}^{\pi} K_N(t)\,dt = \frac{\epsilon}{3},$$

using facts from Lemma 4.2 (2) and (3). Thus, finally, we get

$$|P_N(x) - f(x)| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \qquad x \in [-\pi,\pi],$$

as desired. Note that if f is real valued, then so is $P_N = K_N * f$ by Lemma 4.2 (2). □

A function f (of the kind met in Calculus I) is said to satisfy a *Lipschitz continuity* condition at a number x if there exists positive numbers $M, \delta > 0$ such that

$$|f(y) - f(x)| \le M|y - x|, \qquad \text{whenever } |y - x| < \delta.$$

Theorem 4.4. *Suppose that f is a 2π-periodic function which is Riemann integrable on $[-\pi, \pi]$ and satisfies a Lipschitz continuity condition at a number x. Then the Fourier series of f converges to $f(x)$ at x.*

Proof. The Lipschitz condition at x may be rephrased as: $|f(x - t) - f(x)| \le M|t|$ whenever $|t| < \delta$. For simplicity assume $x \in (-\pi, \pi)$, but it is easy to amend the argument below if x is an endpoint. We are fixing such an $x \in (-\pi, \pi)$ thoughout the proof. Define $g(t)$ to be $\frac{f(x-t)-f(x)}{\sin(t/2)}$ whenever t is not an even multiple of 2π, and is otherwise zero. By Lemma 4.1 (1) we have $(f(x - t) - f(x))D_N(t) = g(t)\sin((N + \frac{1}{2})t)$, and so using Lemma 4.1 (2) and (3) we obtain a convenient expression for $s_N(f)(x) - f(x)$, namely

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\, D_N(t)\, dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, D_N(t)\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\sin((N+\frac{1}{2})t)\, dt.$$

Thus by the trig double angle formula we have

$$s_N(f)(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\cos(t/2)\sin(Nt)\, dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\sin(t/2)\cos(Nt)\, dt.$$

We show using the Riemann-Lebesgue lemma in the previous section that both of these last integrals have limit 0 as $n \to \infty$. For the second integral note that $g(t)\sin(t/2) = f(x - t) - f(x)$, which is Riemann integrable on $[-\pi, \pi]$ so that by the Riemann-Lebesgue lemma its Fourier coefficients converge to 0. Thus $\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\sin(t/2)\cos(Nt)\, dt \to 0$ as $n \to \infty$. For the first integral, rewrite $g(t)\cos(t/2)$ as a product $h(t)\, k(t)$; where $h(t) = \frac{f(x-t)-f(x)}{t}$ and $k(t) = \frac{t}{\tan(t/2)}$ when t is not an even multiple of 2π, and are otherwise 0. On $[-\pi, \pi]$ the function k is Riemann integrable (a Calculus or Math 3333 exercise). Moreover, h is Riemann integrable too as long as we stay away from $t = 0$. That is, h is Riemann integrable on $[-\pi, -\epsilon] \cup [\epsilon, \pi]$ for any $\epsilon > 0$. An easy Math 3333 exercise: if F is a bounded function on $[a, b]$ and F is Riemann integrable on h is Riemann integrable $[a+\epsilon, b]$ for all $\epsilon > 0$, then F is Riemann integrable on $[a, b]$. By this principle, since the Lipschitz condition at x ensures that h is bounded (by M), we see that h is Riemann integrable on $[-\pi, \pi]$. Hence by Math 3333/4331, $g(t)\cos(t/2) = h(t)k(t)$ is Riemann integrable on $[-\pi, \pi]$. Thus, again by the Riemann-Lebesgue lemma its Fourier coefficients converge to 0. Thus $\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\cos(t/2)\sin(Nt)\, dt \to 0$ as $n \to \infty$. □

Corollary 4.5. *If f is a 2π-periodic function which is Riemann integrable on $[-\pi, \pi]$ and is differentiable at a point x, then the Fourier series of f converges to $f(x)$ at x.*

Proof. It is well known and easy to see that differentiable at a point x implies the Lipschitz condition at x. (Hint: look at the ϵ-δ definition of $\lim_{y \to x} \frac{f(y)-f(x)}{y-x} = f'(x)$, with $\epsilon = 1$. See Homework 11 question 6.) So the result follows from Theorem 4.4. □

Corollary 4.6. (Localization theorem) *Suppose that f and g are 2π-periodic functions which are Riemann integrable on $[-\pi, \pi]$, and that $f = g$ on an open interval J. Then if $x \in J$ then the Fourier series of f and g at x either both converge to the same value, or both diverge.*

Proof. Let $h = f - g$, then $h = 0$ on J. So by Corollary 4.5 the Fourier series of h converges to 0 at any $x \in J$. But the Fourier series of h is the Fourier series of f minus the Fourier series of g (see the proof of Corollary 1 in the last section). So we are done (because for numbers z_k, w_k, if $0 = \sum_k (z_k - w_k)$ converges then either $\sum_k z_k$ and $\sum_k w_k$ converge to the same value, or both diverge (Exercise)). □

This result says that the pointwise convergence or divergence of a Fourier series of a function f at x only depends on values of f nearby x, where by 'nearby' we mean in a (small) interval J containing x. Changing f outside of J does not effect whether the Fourier series converges or diverges at x, or its limit if it converges, by Corollary 4.6.

We recall from Chapter 0 that if s_n is the nth partial sum of a series $\sum_{k=0}^{\infty} a_k$ of numbers, then the *Cesáro means* are the sequence (σ_n) defined by

$$\sigma_n = \frac{1}{n+1}(s_0 + s_1 + s_2 + \cdots + s_n).$$

We said that $\sum_k a_k$ is *Cesáro summable* if the sequence (σ_n) converges, and in this case $\lim_n \sigma_n$ is called the *Cesáro sum* of the series $\sum_k a_k$. We proved in Chapter 0 that if $\sum_k a_k$ converges with sum s, then it is Cesáro summable and its Cesáro sum is s.

We apply this notation to the nth partial sum $s_n(f)(x)$ of a Fourier series. Thus we define

$$\sigma_N(f)(x) = \frac{s_0(f)(x) + s_1(f)(x) + s_2(f)(x) + \cdots + s_N(f)(x)}{N+1}.$$

We say that a Fourier series is *Cesáro summable at x* if the sequence $(\sigma_N(f)(x))$ converges as $N \to \infty$, and in this case $\lim_{N \to \infty} \sigma_N(f)(x)$ is called the *Cesáro sum at x* of the Fourier series. By the above fact from Chapter 0, if the Fourier series converges pointwise at x with sum s, then it is Cesáro summable at x and its Cesáro sum at x is s.

Theorem 4.7. (Fejer's theorem) *If f is a 2π-periodic function which is Riemann integrable on $[-\pi, \pi]$, then at any point x where $f(x^-)$ and $f(x^+)$ exist, the Fourier series of f is Cesáro summable at x, and its Cesáro sum at x is* $\lim_{n\to\infty} \sigma_n(f)(x) = \frac{1}{2}(f(x^-) + f(x^+))$.

Proof. This proof is very similar to the proof of Theorem 4.3. In the proof of that result, we said $f * D_k = 2\pi\, s_k(f)$, and hence

$$f*K_N = f*(\frac{1}{N+1}\sum_{k=0}^{N} D_k) = \frac{1}{N+1}\sum_{k=0}^{N} f*D_k = 2\pi\frac{1}{N+1}\sum_{k=0}^{N} s_k(f) = 2\pi\sigma_N(f).$$

Let x be a point such that $f(x^-)$ and $f(x^+)$ exist. Then

$$|\sigma_n(f)(x) - \frac{1}{2}(f(x^-) + f(x^+))| = |\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)\, f(x-t)\, dt - \frac{1}{2}(f(x^-) + f(x^+))|.$$

Write $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)\, f(x-t)\, dt = \frac{1}{2\pi}\int_{0}^{\pi} K_N(t)\, f(x-t)\, dt + \frac{1}{2\pi}\int_{-\pi}^{0} K_N(t)\, f(x-t)\, dt$, and write $f(x^-) = \frac{1}{\pi}\int_0^{\pi} K_N(t)\, f(x^-)\, dt$ and $f(x^+) = \frac{1}{\pi}\int_{-\pi}^{0} K_N(t)\, f(x^+)\, dt$, using Lemma 4.2 (3). So $|\sigma_n(f)(x) - \frac{1}{2}(f(x^-) + f(x^+))|$ is dominated by

$$\frac{1}{2\pi}|\int_0^{\pi} K_N(t)\, f(x-t)\, dt - \int_0^{\pi} K_N(t)\, f(x^-)\, dt| + \frac{1}{2\pi}|\int_{-\pi}^{0} K_N(t)\, f(x-t)\, dt - \int_{-\pi}^{0} K_N(t)\, f(x^+)\, dt|.$$

Given $\epsilon > 0$ there exists $\delta > 0$ such that $|f(y) - f(x^+)| < \frac{\epsilon}{2}$ whenever $x < y < x+\delta$, and such that $|f(y) - f(x^-)| < \frac{\epsilon}{2}$ whenever $x > y > x-\delta$. Since f is bounded there is a constant M dominating $|f|$. By Lemma 4.2 (4), there exists N_0 so that

$$K_N(x) \leq \frac{2}{(N+1)(1-\cos\delta)} < \frac{\epsilon}{4M}, \qquad 0 < \delta \leq |x| \leq \pi,\ N \geq N_0.$$

Now $\frac{1}{2\pi}\int_0^{\pi} K_N(t)\, |f(x-t) - f(x^-)|\, dt$ is dominated by

$$\frac{1}{2\pi}\int_0^{\delta} K_N(t)\, |f(x-t) - f(x^-)|\, dt + \frac{1}{2\pi}\int_{\delta}^{\pi} 2M \cdot \frac{\epsilon}{4M} dt < \frac{1}{2\pi}\int_0^{\delta} K_N(t)\, \frac{\epsilon}{2} dt + \frac{\epsilon}{4}$$

for $N \geq N_0$. Similarly,

$$\frac{1}{2\pi}\int_{-\pi}^{0} K_N(t)\, |f(x-t) - f(x^+)|\, dt < \frac{1}{2\pi}\int_{-\delta}^{0} K_N(t)\, \frac{\epsilon}{2} dt + \frac{\epsilon}{4}, \qquad N \geq N_0,$$

so that finally,

$$|\sigma_n(f)(x) - \frac{1}{2}(f(x^-) + f(x^+))| \leq \frac{1}{2\pi}\int_{-\delta}^{\delta} K_N(t)\, \frac{\epsilon}{2} dt + \frac{\epsilon}{4} + \frac{\epsilon}{4} \leq \epsilon$$

for $N \geq N_0$. □

Remark. It is known that a Riemann integrable function is continuous except on a set of Lebesgue measure zero. Thus Fejer's theorem implies that if f is a 2π-periodic function which is Riemann integrable on $[-\pi, \pi]$, then $\lim_{n\to\infty} \sigma_n(f)(x) = f(x)$ for all x outside a set of Lebesgue measure zero. Carleson's theorem implies that one can replace σ_n by s_n in the last statement, but that is an incredibly deep result.

Corollary 4.8. *If f is a continuous 2π-periodic function then $\sigma_n(f) \to f$ uniformly.*

Proof. Notice that by Math 3333, f is uniformly continuous on $[-\pi, \pi]$, so that in the proof of Fejer's theorem above $f(x^+) = f(x^-) = f(x)$ and δ is independent of x. The inequalities at the end of that proof then do not depend on x, so that $\sigma_n(f)(x) \to f$ uniformly. □

Corollary 4.9. *If f is a 2π-periodic function which is Riemann integrable on $[-\pi, \pi]$, and if the Fourier series for f converges pointwise at a point x, and if $f(x^-)$ and $f(x^+)$ exist, then the Fourier series for f converges pointwise at x to $\frac{1}{2}(f(x^-) + f(x^+))$.*

Proof. This follows from Fejer's theorem above together with the fact mentioned before Fejer's theorem that if the Fourier series converges pointwise at x with sum s, then it is Cesáro summable at x and its Cesáro sum at x is s. However by Fejer's theorem its Cesáro sum at x is $\frac{1}{2}(f(x^-) + f(x^+))$, so the Fourier series for f converges pointwise at x to $\frac{1}{2}(f(x^-) + f(x^+))$. □

Remark. Putting Corollary 4.9 together with Corollary 3 at the end of the last section, one sees that if f is a 2π-periodic function which is Riemann integrable on $[-\pi, \pi]$, and if the sum of the Fourier coefficients for f converges absolutely, then the Fourier series for f converges pointwise at x to $\frac{1}{2}(f(x^-) + f(x^+))$ for every x for which $f(x^-)$ and $f(x^+)$ exist.

Corollary 4.10. *If f is a 2π-periodic function which is Riemann integrable on $[-\pi, \pi]$, and if n times the nth Fourier coefficients of f (for all integers n) constitute a bounded set, then the Fourier series for f converges pointwise at x to $\frac{1}{2}(f(x^-) + f(x^+))$ for every x for which $f(x^-)$ and $f(x^+)$ exist.*

Proof. Let $s = \frac{1}{2}(f(x^-) + f(x^+))$ for an x where these limits exist. Let $s_n = s_n(f)(x)$, then $|n(s_n - s_{n-1})| = n|c_n e^{inx} + c_{-n} e^{-inx}| \leq n|c_n| + n|c_{-n}|$, which is bounded by hypothesis. By Fejer's theorem (s_n) is Cesáro summable with Cesáro sum s. So by the Fact in Homework 11 Question 5, $s_n(f)(x) \to s$ as $n \to \infty$. □

A nice application of Corollary 4.10 is the following result, which we will not prove for lack of time, although it is not hard to prove. Energetic graduate students could try prove it as an exercise using Corollary 4.10, or look up its proof.

Corollary 4.11. *If f is a 2π-periodic function which is Riemann integrable on $[-\pi, \pi]$, and if f is of bounded variation on an interval $[\alpha, \beta]$, then the Fourier series for f converges pointwise to $\frac{1}{2}(f(x^-) + f(x^+))$ for all $x \in [\alpha, \beta]$.*

The following is mostly in the homework (and also appears on the Mock test 2). It could very well be on the real test since it is homework (some parts are very close to the homework).

In some of the homework, and in the example below, one might have to integrate a Fourier series. This works because we saw in the first theorem stated in this pdf that every Fourier series of a Riemann integrable function converges in 2-norm, and because of the following:

Proposition 4.12. *Suppose we have a sequence of functions $h, h_1, h_2, h_3, \cdots$ which are Riemann integrable on $[a, b]$, and $a \leq c < d \leq b$, then. Then*

1. *If $h_n \to h$ in 2-norm on $[a, b]$ then $\int_c^d h_n \, dx \to \int_c^d h \, dx$.*
2. *If $\sum_{k=1}^\infty h_k = h$ in 2-norm, then $\int_c^d h \, dx = \sum_{k=1}^\infty \int_c^d h_n \, dx$.*

Proof. (1) We have

$$|\int_c^d h_n \, dx - \int_c^d h \, dx| \leq \int_c^d |h_n - h| \, dx \leq \int_a^b 1 \cdot |h_n - h| \, dx \leq \|1\|_2 \|h_n - h\|_2 \to 0,$$

where we have used the Cauchy-Schwarz inequality for integrals.

(2) If $s_n = \sum_{k=1}^n h_k$ then $s_n \to h$ in 2-norm, so by (1) applied to s_n we have $\int_c^d s_n \, dx = \sum_{k=1}^n \int_c^d h_k \, dx \to \int_c^d h \, dx$. Thus $\sum_{k=1}^\infty \int_c^d h_n \, dx = \int_c^d h \, dx$. □

Example.

(a) Show that $x = \pi - 2\sum_{n=1}^\infty \frac{\sin nx}{n}$ if $0 < x < 2\pi$.
(b) Using Parseval's equation in (a) find $\sum_{n=1}^\infty \frac{1}{n^2}$.
(c) Using (a) and (b) show that $\frac{x^2}{2} = \pi x - \frac{\pi^2}{3} + 2\sum_{n=1}^\infty \frac{\cos nx}{n^2}$ if $0 \leq x \leq 2\pi$.
(d) Using (c) find $\sum_{n=1}^\infty \frac{(-1)^n}{n^2}$.

Solution. (a) Make the function x on $[0, 2\pi)$ to be periodic of period 2π by just repeating it endlessly. Call this 2π-periodic function f. So e.g. $f(x) = x + 2\pi$ for $-\pi < x < 0$. The Fourier series on $[-\pi, \pi]$ is easy to compute: it is $\pi - 2\sum_{n=1}^\infty \frac{\sin nx}{n}$. By one (or several) of the Theorems or Corollaries above (eg. Corollary 4.5), this series converges pointwise to $f(x)$ on $(0, \pi]$ and on $[-\pi, 0)$, hence by periodicity on $[\pi, 2\pi)$. Thus $x = \pi - 2\sum_{n=1}^\infty \frac{\sin nx}{n}$ if $0 < x < 2\pi$.

(b) By Parseval's equation applied in (a), $\int_{-\pi}^\pi |f(x)|^2 \, dx = \pi(2\pi^2 + 4\sum_{n=1}^\infty \frac{1}{n^2}$. That is, $\int_0^{2\pi} x^2 \, dx = \frac{8\pi^3}{3} = \pi(2\pi^2 + 4\sum_{n=1}^\infty \frac{1}{n^2}$, so that $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$.

(c) If $0 \leq x \leq 2\pi$ then integrate $\int_0^x$ in (a). We have to use the fact just above the Example to integrate the series in (a). We get

$$\frac{x^2}{2} = \pi x - \int_0^x (2\sum_{n=1}^\infty \frac{\sin nx}{n}) dx = \pi x - 2\sum_{n=1}^\infty \int_0^x \frac{\sin nx}{n} \, dx = \pi x - 2\sum_{n=1}^\infty (\frac{\cos nx}{n^2} - \frac{1}{n^2}).$$

But $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$. So $\frac{x^2}{2} = \pi x - \frac{\pi^2}{3} + 2\sum_{n=1}^\infty \frac{\cos nx}{n^2}$ if $0 \leq x \leq 2\pi$.

(d) Set $x = \pi$ in (c) to get $\frac{\pi^2}{2} = \pi^2 - \frac{\pi^2}{3} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$. So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$.