INTRODUCTION TO REAL ANALYSIS II
MATH 4332–BLECHER NOTES

You will be expected to reread and digest these typed notes after class, line by line, trying to follow why the line is true, for example how it follows from previous lines. I suggest you add a check mark after you have read and understood the line, add extra explanation or pictures to yourself if needed. Add a question mark next to any line you cannot follow, and ask me or the TA about it. That is why I have given wide margins on every page. Also memorize 'definitions' as you read. The best advice I can give to ensure success in this class is to do this reading properly. In my experience, the class becomes much much more difficult if you do not do it. This kind of detailed reading is not without pain, but it will help reconfigure your brain to internalize the kind of logic and proofs that are needed in this subject (and in other math 'proof' courses). The way I will monitor if you are doing all this, and this is the first semester I've tried this, is to collect your notes occasionally to see how much you have scribbled on them as above. There will also be an 'easy-quiz' from time to time, named as such because it will be easy for anyone who has been reading the notes as suggested.

The problems marked * do not have to be turned in by the undergraduates in the class.

Chapter 0. Series of numbers

In this course/these notes we will write $\mathbb{N}$ for the *natural numbers* $\mathbb{N} = \{1, 2, 3, \cdots\}$, and $\mathbb{N}_0$ for the *whole numbers* $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$, and $\mathbb{Z}$ for the *integers* $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, 3, \cdots\}$. We usually reserve the symbols n, m for natural numbers or integers. The *rational numbers* are $\mathbb{Q} = \{m/n : m, n \in \mathbb{Z}, n \neq 0\}$. The *real numbers* are written as $\mathbb{R}$, and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$.

The notation $A \subseteq B$ or $A \subset B$ means that A is a subset of set B, (that is, $x \in A \Rightarrow x \in B$). Of course $A = B$ if $A \subset B$ and $B \subset A$ (that is, $x \in A \Leftrightarrow x \in B$). Complement: $A^c = \{x : x \notin A\}$. We say that sets A and B are *disjoint* if $A \cap B = \emptyset$ (no common elements).

Functions: $f : A \to B$ means that f is a function from domain A <u>into</u> the codomain B. Image: if $C \subseteq A$, and $f : A \to B$, then $f(C) = \{f(x) : x \in C\}$. This is a subset of the codomain of f, called the *image* of C under f. Pre-image: If $f : A \to B$, and $D \subseteq B$, then $f^{-1}(D) = \{x \in A : f(x) \in D\}$. This is called the *pre-image* of D under f. We say that $f : A \to B$ is *surjective*, or *onto* if $f(A) = B$.

We say that $f : A \to B$ is *injective*, or *one-to-one* if $f(x_1) = f(x_2) \;\Rightarrow\; x_1 = x_2$. We say that $f : A \to B$ is *bijective* if it is one-to-one and onto. In this case there is an inverse function $f^{-1} : B \to A$ with $f^{-1}(y) = x$ iff $f(x) = y$, for $x \in A, y \in B$.

1. Limsup and liminf

See wikipedia for more detail.

Definition. The *limit superior* of a sequence (s_n), is the number

$$\text{limsup}_n\, s_n = \lim_{n\to\infty} \{\sup\{s_k : k \geq n\}\}.$$

Sometimes this is written as $\overline{\lim}_n\, s_n$. The *limit inferior* is

$$\text{liminf}_n\, s_n = \lim_{n\to\infty} \{\inf\{s_k : k \geq n\}\}.$$

Sometimes this is written as $\underline{\lim}_n\, s_n$.

Ex. Find the limsup and liminf of (s_n) where $s_n = (-1)^n + \frac{1}{n}$.

Solution. Make a table of the terms in the sequence:

$$0, \frac{3}{2}, -\frac{2}{3}, \frac{5}{4}, -\frac{4}{5}, \frac{7}{6}, -\frac{6}{7}, \frac{9}{8}, \cdots .$$

Procedure: explained in class to get the new row in the table

$$\frac{3}{2}, \frac{3}{2}, \frac{5}{4}, \frac{5}{4}, \frac{7}{6}, \frac{7}{6}, \frac{9}{8}, \cdots .$$

(Each entry in this row is the supremum of the numbers above and to the right of it in the original row.)

Note that this new row is a decreasing sequence. Every decreasing sequence has a limit (which is possibly $\pm\infty$). In our example, this limit is 1. This limit is exactly $\text{limsup}_n\, s_n$. So in our example, $\text{limsup}_n\, s_n = 1$.

Similarly to find the liminf, make a third row of the table

$$-1, -1, -1, -1, -1, \cdots .$$

(Each entry in this row is the infimum of the numbers above and to the right of it in the original row.)

Note that this third row is always an increasing sequence. Every increasing sequence has a limit (which is possibly $\pm\infty$). In our example, this limit is -1. This limit is exactly $\text{liminf}_n\, s_n$. So in our example, $\text{liminf}_n\, s_n = -1$.

Ex. Find the limsup and liminf of (s_n) where $s_n = \frac{6n+4}{7n-3}$.

Solution. Do this as an exercise.

What the limsup and liminf are good for:

- First, they always exist, unlike the limit. For example, in the first example above, $\lim_n s_n$ does not exist. But we were able to compute the limsup and liminf. They always exist because as we saw in the example, they are limits of monotone sequences, which we know always exist.
- They behave similarly to the limit. That is, they obey laws analogous to the rules we saw in Calculus II and 3333 for limits. We will write down some of these laws momentarily.
- They can be used to check if the limit exists. In fact $\lim_n s_n$ exists iff $\liminf_n s_n = \limsup_n s_n$. So if $\liminf_n s_n \neq \limsup_n s_n$ then we may conclude that $\lim_n s_n$ does not exist.
- Recall that in Calculus II there were certain tests which involve the limit of a sequence, such as the ratio and root test, limit comparison test, and the 'fundamental fact about power series'. We shall see that one can improve these tests by using the limsup and liminf instead of the limit. For example, the 'fundamental fact about power series' states that a power series $\sum_{k=0}^{\infty} c_k x^k$ converges absolutely for all points in a certain interval $(-R, R)$, and it diverges whenever $x < -R$ or $x > R$. The number R is called the *radius of convergence*, and in Calculus II you are given the formula $R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{c_n}}$. But you were not told in Calculus II what to do if this limit does not exist. In fact one should use the formula $R = \frac{1}{\limsup_n \sqrt[n]{c_n}}$. This always exists, as we remarked above, and now the test always works.

Theorem 1.1. *Let (s_n) and (t_n) be sequences of real numbers.*

(1) $\liminf_n s_n \leq \limsup_n s_n$; *and* $\lim_n s_n$ *exists iff* $\liminf_n s_n = \limsup_n s_n$.
(2) $\liminf_n (-s_n) = -\limsup_n s_n$.
(3) *If* $s_n \leq t_n$ *for all* n, *then* $\limsup_n s_n \leq \limsup_n t_n$, *and* $\liminf_n s_n \leq \liminf_n t_n$.
(4) $\limsup_n (K s_n) = K \limsup_n s_n$, *and* $\liminf_n (K s_n) = K \liminf_n s_n$, *if* $K \geq 0$.
(5) $\limsup_n (s_n + t_n) \leq \limsup_n s_n + \limsup_n t_n$, *and* $\liminf_n (s_n + t_n) \geq \liminf_n s_n + \liminf_n t_n$. *These inequalities are equalities if* (t_n) *converges.*
(6) *If* $a = \limsup_n s_n$ *is finite, and if* $\epsilon > 0$, *then there exists an* $N \geq 1$ *such that* $s_n < a + \epsilon$ *for all* $n \geq N$. *Also for every* $N \geq 1$ *there exists a* $k > N$ *with* $s_k > a - \epsilon$. *That is, there are infinitely many terms in the sequence which are greater than* $a - \epsilon$. *(We remark that these properties actually characterize* $\limsup_n s_n$; *and variants of these statements are easily formulated for liminfs.)*
(7) *If* $a = \limsup_n s_n$ *then a subsequence of* (s_n) *converges to* a. *Similarly for liminf.*

Proof. (1) Clearly $\inf\{s_k : k \geq n\} \leq \sup\{s_k : k \geq n\}$, and so $\lim_{n\to\infty} \inf\{s_k : k \geq n\} \leq \lim_{n\to\infty} \sup\{s_k : k \geq n\}$. We also see from this that if these numbers are finite then $\text{liminf}_n\, s_n = \text{limsup}_n\, s_n$ iff

$$0 = \text{limsup}_n\, s_n - \text{liminf}_n\, s_n = \lim_{n\to\infty} (\sup\{s_k : k \geq n\} - \inf\{s_k : k \geq n\}).$$

By the definition of limit, this happens iff given $\epsilon > 0$ there exists an N such that

$$\sup\{s_k : k \geq n\} - \inf\{s_k : k \geq n\} \leq \epsilon, \qquad n \geq N.$$

But the latter is equivalent to

$$|s_m - s_n| \leq \epsilon, \qquad m, n \geq N.$$

This is saying (s_n) is Cauchy, which is equivalent to saying that $\lim_n\, s_n$ exists. If $\text{liminf}_n\, s_n = \text{limsup}_n\, s_n = \infty$, then since $\inf\{s_k : k \geq n\} \leq s_n$ we see $\lim_n\, s_n = \infty$. Similarly for the $-\infty$ case.

(2) From 3333 we have $\inf\{-s_k : k \geq n\} = -\sup\{s_k : k \geq n\}$. Taking the limit of these as $n \to \infty$ we get $\text{liminf}_n\,(-s_n) = -\text{limsup}_n\, s_n$.

(3) $\sup\{s_k : k \geq n\} \leq \sup\{t_k : k \geq n\}$. Taking the limit of these as $n \to \infty$ we get $\text{limsup}_n\, s_n \leq \text{limsup}_n\, t_n$.

(4) From 3333 we have $\sup\{Ks_k : k \geq n\} = K \sup\{s_k : k \geq n\}$. Taking the limit of these as $n \to \infty$ we get $\text{limsup}_n\,(Ks_n) = K\, \text{limsup}_n\, s_n$.

(5) From 3333 we have $\sup\{s_k + t_k : k \geq n\} \leq \sup\{s_k : k \geq n\} + \sup\{t_k : k \geq n\}$. Taking the limit of these as $n \to \infty$ we get $\text{limsup}_n\,(s_n + t_n) \leq \text{limsup}_n\, s_n + \text{limsup}_n\, t_n$. Similarly for the liminf case. If (t_n) converges to $t \in \mathbb{R}$ then given $\epsilon > 0$ there exists an $N \geq 1$ such that $t_n > t - \epsilon$ for all $n \geq N$. Then

$$\sup\{s_k + t_k : k \geq n\} \geq \sup\{s_k + t - \epsilon : k \geq n\} = \sup\{s_k : k \geq n\} + t - \epsilon, i$$

for $n \geq N$. Taking the limit of these as $n \to \infty$ we get $\text{limsup}_n\,(s_n + t_n) \geq \text{limsup}_n\, s_n + \lim_n t_n - \epsilon$, for all $\epsilon > 0$. So $\text{limsup}_n\,(s_n + t_n) = \text{limsup}_n\, s_n + \text{limsup}_n\, t_n$.

(6) If $a = \text{limsup}_n\, s_n = \lim_{n\to\infty} \sup\{s_k : k \geq n\}$ is finite then from 3333 given $\epsilon > 0$ there exists an $N \geq 1$ such that $a - \epsilon < \sup\{s_k : k \geq n\} < a + \epsilon$ for all $n \geq N$. So $s_n \leq \sup\{s_k : k \geq n\} < a + \epsilon$ for all $n \geq N$. This proves the first assertion. Similarly, $\sup\{s_k : k \geq n\} > a - \epsilon$ for all $n \geq N$, which means that there must exists a $k > n$ with $s_k > a - \epsilon$.

(7) If we take $\epsilon = \frac{1}{n}$ in the argument in (6), we see that given any $m \geq N$ we have $a - \frac{1}{n} < \sup\{s_k : k \geq m\} < a + \frac{1}{n}$. So given any $m \geq N$ there exists a $k \geq m$ with $|s_k - a| < \frac{1}{n}$. Applying this with $n = 1, m = N$, choose n_1 with $|s_{n_1} - a| < 1$. Applying this with $n = 2$ and $m > n_1$, choose $n_2 > n_1$ with $|s_{n_2} - a| < \frac{1}{2}$. Applying this again with $n = 3$ and $m > n_2$, choose $n_3 > n_2$ with $|s_{n_3} - a| < \frac{1}{3}$. Continuing in this way produces a subsequence of (s_n) converging to a.

The proofs of (3)–(7) above in the liminf cases are similar. □

(The last theorem used to be Homework 1, due Thursday 22 January. The grading scale was Q1 [2 points for first part, 9 for second part], Q2 [3 points], Q4 [2 points], Q6 [6 points for first part, 3 for second part]. Total [25] points for correctness, [5] points for completeness in Q 3, 5, 7.)

2. Infinite series of numbers

The following from Calculus II is almost all review, so we will move quickly. You may need to read it carefully several times. You could also look up several of these topics on wikipedia.

From Calculus II: An 'infinite series' is an expression of the form

$$\sum_{k=m}^{\infty} a_k \;=\; a_m + a_{m+1} + a_{m+2} + \cdots \qquad (*)$$

Let us call this expression (*). The a_k here are real (or complex) numbers.

What does expression (*) mean? In fact we shall see shortly that the expression means two things.

Usually $m = 0$ or 1, that is, (*) usually is

$$a_0 + a_1 + a_2 + \cdots$$

or

$$a_1 + a_2 + a_3 + \cdots .$$

We call the number a_k the *kth term in the series.* Sometimes we will be sloppy and write $\sum_k a_k$ when we mean (*).

The most important question about an infinite series, just as for an infinite sequence, is 1) does the series converge? and 2) if it converges, what is its sum? We will explain these in a minute.

In fact an expression like (*) has two meanings:

Meaning # 1: A 'formal sum'. That is, it is a way to indicate that we are thinking about adding up all these numbers in the expression (*), in the order given. It does not mean that these numbers do add up.

Before we go to Meaning # 2, let me say how you 'add up all the numbers in an infinite series'. To do this, we define the *nth partial sum* s_n to be the sum of the first n terms in the series. In this way we get a *sequence*

$$s_1, s_2, s_3, \cdots$$

called the *sequence of partial sums.* For example, for the series $\sum_{k=0}^{\infty} a_k$, we have $s_n = \sum_{k=0}^{n-1} a_k$. We say the original series *converges* if the *sequence* $\{s_n\}$ converges. If it does not converge then we say it *diverges.* We call $\lim_{n\to\infty} s_n$ the *sum of the series* if this limit exists.

Meaning # 2: $\sum_k a_k = \lim_{n\to\infty} s_n$ if this limit exists.

- Cauchy test: $\sum_k a_k$ converges iff given $\epsilon > 0$ there exists an $N \geq 0$ such that $|a_{n+1} + a_{n+2} + \cdots + a_m| < \epsilon$ whenever $m > n \geq N$.

  [Proof: Since $s_m - s_n = a_{n+1} + a_{n+2} + \cdots + a_m$, this is just saying that the partial sums $s_n = \sum_{k=1}^n a_k$ are a Cauchy sequence. And we know from Math 3333 that a sequence converges iff it is a Cauchy sequence.]

- In a sum like $\sum_{k=1}^\infty \frac{1}{(k+1)k}$, the k is a 'dummy index'. That is, it is only used internally inside the sum, and we can feel free to change its name, to $\sum_{j=1}^\infty \frac{1}{(j+1)j}$, for example,

- In a series $\sum_{k=m}^\infty a_k$ let us call m the 'starting index'. Thus for example, the starting index of $\sum_{k=2}^\infty \frac{k-1}{k^2}$ is 2. Any series can be 'renumbered' so that its starting index is 0. That is, any infinite series may be rewritten as $\sum_{k=0}^\infty a_k$. For example, $\sum_{k=m}^\infty a_k$, which is the same as $a_m + a_{m+1} + a_{m+2} + \cdots$, can be relabelled by letting $j = k - m$, or equivalently $k = j + m$. Then $\sum_{k=m}^\infty a_k = \sum_{j=0}^\infty a_{j+m}$.

Example: Rewrite $\sum_{k=2}^\infty \frac{k-1}{k^2}$ as a series $\sum_{k=0}^\infty a_k$.

Solution. Letting $j = k - 2$, so that $k = j + 2$, the sum becomes

$$\sum_{j=0}^\infty \frac{j+2-1}{(j+2)^2} = \sum_{j=0}^\infty \frac{j+1}{(j+2)^2}$$

Of course j is 'dummy' so we can rewrite this as $\sum_{k=0}^\infty \frac{k+1}{(k+2)^2}$.

There is no reason of course why we chose 0 for the starting index. One can make all series begin with the starting index 1 if you wanted to, by a similar trick. However it is convenient to fix one starting index, so it may as well be 0. Many of the following results are therfore phrased in terms of series $\sum_{k=0}^\infty a_k$.

- **Geometric series:** This is a series of form $c + cx + cx^2 + cx^3 + \cdots$, or $\sum_{k=0}^\infty cx^k$, for constants c and x. We call x the 'constant ratio' of the geometric series. Note that if you divide any term in the series by the previous term, you get x. We assume $c \neq 0$, otherwise this is the trivial series with sum 0.

  The MAIN FACT about geometric series, is that such a series converges if and only if $|x| < 1$, and in that case its sum is $\frac{c}{1-x}$.

[Proof: If $c = 0$ then the series is $0 + 0 + 0 + \cdots$ and the result is obvious. So we can assume that $c \neq 0$. The sum of the first n terms is

$$s_n = c + cx + cx^2 + \cdots + cx^{n-1} = c(1 + x + x^2 + \cdots + x^{n-1}) \ .$$

There is a well known result in algebra which says that

$$(1 + x + x^2 + \cdots + x^{n-1})(1 - x) = 1 - x^n$$

(to prove it multiply out the parentheses and cancel). Thus if $x \neq 1$ then $1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$, so that

$$s_n = c + cx + cx^2 + \cdots + cx^{n-1} = c\frac{1 - x^n}{1 - x} \ .$$

This is the important formula for the sum of n terms of a geometric series. The only thing that depends on n on the right hand side here is the x^n, which converges to 0 if $|x| < 1$, and diverges otherwise. If $x = 1$ then $s_n = c + c + \cdots + c$ (n times) which equals nc. Thus $\lim_{n\to\infty} s_n = c\frac{1}{1-x}$ if $|x| < 1$. If $|x| \geq 1$ then $\{s_n\}$ diverges, so that the original series diverges.]

- FACT: If $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ both converge, and if c is a constant, then:
- $\sum_{k=0}^{\infty} (a_k + b_k)$ converges, with sum $\sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$;
- $\sum_{k=0}^{\infty} (a_k - b_k)$ converges, with sum $\sum_{k=0}^{\infty} a_k - \sum_{k=0}^{\infty} b_k$;
- $\sum_{k=0}^{\infty} (ca_k)$ converges, with sum $c\sum_{k=0}^{\infty} a_k$.

[Proof: We just prove the first and third, the second is quite similar. The nth partial sum of $\sum_{k=0}^{\infty} (a_k + b_k)$ is $\sum_{k=0}^{n-1} (a_k + b_k) = \sum_{k=0}^{n-1} a_k + \sum_{k=0}^{n-1} b_k$. By a fact about sums of limits of *sequences* from 3333 or 4331, this converges, as $n \to \infty$, to $\sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$.

Similarly the nth partial sum of $\sum_{k=0}^{\infty} (ca_k)$ is $\sum_{k=0}^{n-1} (ca_k) = c\sum_{k=0}^{n-1} a_k$. By 3333, this converges, as $n \to \infty$, to $c\sum_{k=0}^{\infty} a_k$.]

- For any positive integer m we can write $\sum_{k=0}^{\infty} a_k = (a_0 + a_1 + \cdots + a_{m-1}) + \sum_{k=m}^{\infty} a_k$.

  Indeed $\sum_{k=0}^{\infty} a_k$ converges if and only if $\sum_{k=m}^{\infty} a_k$ converges. If these series converge, then their sum also obeys the rule:

$$\sum_{k=0}^{\infty} a_k = (a_0 + a_1 + \cdots + a_{m-1}) + \sum_{k=m}^{\infty} a_k \ .$$

[This is because the nth partial sum of the $\sum_{k=0}^{\infty} a_k$ series, and the nth partial sum of the $\sum_{k=m}^{\infty} a_k$ series differ by a fixed constant, namely $a_0 + a_1 + \cdots + a_{m-1}$.]

- From the last fact it follows that the 'first few terms' of a series, do not affect whether the series converges or not. It will affect the sum though.

- **The Divergence Test:** If $\lim_{k\to\infty} a_k \neq 0$ then the series $\sum_k a_k$ diverges.

  A matching statement (the contrapositive): If $\sum_k a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

  [Beware: If $\lim_{k\to\infty} a_k = 0$ we cannot conclude that $\sum_k a_k$ converges.

  [Proof: Suppose that $\sum_{k=0}^{\infty} a_k = s$. If s_n is the nth partial sum then $s_n \to s$ as $n \to \infty$. Clearly $s_{n+1} \to s$ too, as $n \to \infty$. Thus $a_n = s_{n+1} - s_n \to s - s = 0$.]

Homework 2 (due Tuesday 27 January).

(1) If $\sum_k a_k$ converges define the *tail* of the series to be the sequence whose nth term is $\sum_{k=n}^{\infty} a_k$. Prove the tail converges to 0 as $n \to \infty$.

(2) If $y \in \mathbb{R}$ write $[y] = \max\{n \in \mathbb{Z} : n \leq y\}$. If $x \in [0,1)$ write $a_1 = [10x], a_2 = [100(x - \frac{a_1}{10})], a_3 = [1000(x - \frac{a_1}{10} - \frac{a_2}{100})], \cdots$. Prove that $0 \leq a_k \leq 9$ for each k and that $\sum_{n=1}^{\infty} \frac{a_n}{10^n}$ converges, Prove that $x = \sum_{n=1}^{\infty} \frac{a_n}{10^n}$. Prove that there is no N such that $a_k = 9$ for all $k \geq N$.

(3) Continuing with the last question, if $x = \sum_{n=1}^{\infty} \frac{a_n}{10^n}$ where $a_n \in \{0, 1, \cdots, 9\}$ then we write $x = 0.a_1a_2a_3\cdots$ and call this a *decimal expansion* of x (or base 10 expansion). Prove that the decimal expansion of $x \in [0,1)$ is unique provided that we insist that there is no N such that $a_k = 9$ for all $k \geq N$.

(4) Prove that any nonzero real number may be written as $\pm \sum_{n=N}^{\infty} \frac{a_n}{10^n}$ for some integer N, and with $a_n \in \{0, 1, \cdots, 9\}$ for all n, and $a_N \neq 0$; and that this representation is unique provided that we keep the 'recurring 9 convention' in question (2). [Remark: Questions 2–4 can be found in many places on the internet under 'decimal expansion'; or see e.g. the Appendix B to Tao's Analysis I. It can also be done for any 'base', not just base 10, with almost identical proofs.]

(5)* Suppose $X_1, X_2, \ldots$ are metric spaces, suppose d_n is the metric on X_n, and that $d_n(x,y) \leq 1$ for all $x, y \in X_n$. On the product $\prod_{n=1}^{\infty} X_n$ define $d(\{x_n\}, \{y_n\}) = \sum_{n=1}^{\infty} 2^{-n} d_n(x_n, y_n)$, for $\{x_n\}, \{y_n\} \in \prod_{n=1}^{\infty} X_n$.

  (a) Prove that d is a metric on $\prod_{n=1}^{\infty} X_n$,

  (b) Prove that if each X_n is complete, then so is $\prod_{n=1}^{\infty} X_n$ (with the metric d.)

3. Nonnegative Series, and tests for series convergence.

- A series $\sum_k a_k$ is called a *nonnegative series* if all the terms a_k are ≥ 0.

- For a nonnegative series, the sequence $\{s_n\}$ of the partial sums is a nondecreasing (or increasing) sequence. Indeed if $s_n = a_0 + a_1 + \cdots + a_{n-1}$ say, then $s_{n+1} = a_0 + a_1 + \cdots + a_{n-1} + a_n$, so that $s_{n+1} - s_n = a_n \geq 0$.

  Therefore, by a fact we saw in 3333 for monotone sequences, the sum of the series equals the least upper bound of the sequence $\{s_n\}$ of partial sums. Thus the sum of the series always exists, but may be ∞.

  More importantly, a nonnegative series converges if and only if the $\{s_n\}$ sequence is bounded above. The latter happens if and only if the sum of the series is finite. Thus to indicate that a nonnegative series converges we often simply write $\sum_k a_k < \infty$.

- **Example:** The HARMONIC SERIES is the important series
$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k} .$$
  This is a nonnegative series, so to see if it converges we need only check if the sequence $\{s_n\}$ is bounded above, where $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}$. A trick to do this is to look at $\int_1^{n+1} \frac{1}{x}\, dx$, interpreted as the shaded area in the graph below [Picture drawn in class]. This shaded area is less than the area of the n rectangles shown. Hence
$$1 + 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{3} + \cdots 1 \cdot \frac{1}{n} \geq \int_1^{n+1} \frac{1}{x}\, dx.$$
  So $s_n \geq \ln(n+1) - \ln(1) \to \infty$ as $n \to \infty$.
  Thus the harmonic series diverges; it has sum $+\infty$.

- The trick used in the previous example can be used in the same way to prove:

  The Integral Test: If $f(x)$ is a continuous decreasing positive function defined on $[1, \infty)$ [Picture drawn in class], then $\sum_{k=1}^{\infty} f(k)$ converges if and only if $\int_1^{\infty} f(x)\, dx$ converges (i.e. is finite).

- **p-series.** An almost identical argument shows that $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if and only if $p > 1$. These are called 'p-series'.

- **Basic Comparison Test:** Suppose that $0 \leq a_k \leq b_k$ for all k.
  1) If $\sum_k b_k$ converges, then $\sum_k a_k$ converges.
  2) If $\sum_k a_k$ diverges, then $\sum_k b_k$ diverges.

  [Proof: We have $\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k$. So for 1), if $(\sum_{k=1}^{n} b_k)$ is bounded above then $(\sum_{k=1}^{n} a_k)$ is bounded above. That is, by the third 'bullet' in this section, if $\sum_k b_k$ converges, then $\sum_k a_k$ converges.

Note that 2) is the contrapositive to 1).]

- **Limit Comparison Test:** Suppose that $\sum_k a_k$ and $\sum_k b_k$ are nonnegative series. If $\limsup_{k\to\infty} \frac{a_k}{b_k} < \infty$, and if $\sum_k b_k$ converges then $\sum_k a_k$ converges. If $\liminf_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_k b_k$ diverges then $\sum_k a_k$ diverges.

  [Proof: Suppose that $s = \limsup_{k\to\infty} \frac{a_k}{b_k} < \infty$. By (6) in Homework 1 there exists N with $\frac{a_k}{b_k} < s+1$ for $k \geq N$. So $a_k < (s+1)b_k$ for $k \geq N$. By the Basic Comparison Test, if $\sum_k b_k$ converges then $\sum_k a_k$ converges.

  If $s > 0$ then by the liminf variant of (6) in Homework 1 there exists N with $\frac{a_k}{b_k} > s - \frac{s}{2} = \frac{s}{2}$ for $k \geq N$. So $a_k > \frac{s}{2} b_k$ for $k \geq N$. By the Basic Comparison Test, if $\sum_k b_k$ diverges then $\sum_k a_k$ diverges.]

- **Root Test:** Suppose that $\sum_k a_k$ is a nonnegative series with $\limsup_{k\to\infty} (a_k)^{\frac{1}{k}} = r$. If $0 \leq r < 1$ then $\sum_k a_k$ converges. If $1 < r \leq \infty$ then $\sum_k a_k$ diverges.

  [Proof: Suppose that $r < c < 1$. By (6) in Homework 1 (with $\epsilon = c - r$) there exists N with $a_k^{\frac{1}{k}} < c$ for $k \geq N$. So $a_k < c^k$. Now $\sum_k c^k$ converges (geometric series), so by the Basic Comparison Test, $\sum_k a_k$ converges.

  If $1 < c < r$ then by (6) in Homework 1 there are infinitely many k with $a_k^{\frac{1}{k}} > c$, or equivalently, $a_k > c^k > 1$. So $\sum_k a_k$ diverges by the Divergence Test.]

- **Ratio Test:** Suppose that $\sum_k a_k$ is a nonnegative series with $\limsup_{k\to\infty} \frac{a_{k+1}}{a_k} = R$ and $\liminf_{k\to\infty} \frac{a_{k+1}}{a_k} = r$. If $0 \leq R < 1$ then $\sum_k a_k$ converges. If $1 < r \leq \infty$ then $\sum_k a_k$ diverges.

  [Proof: Suppose that $R < c < 1$. By (6) in Homework 1 (with $\epsilon = c - r$) there exists N with $\frac{a_{k+1}}{a_k} < c$, for $k \geq N$. So $a_{N+1} < ca_N$, $a_{N+2} < ca_{N+1} < c^2 a_N$, etc. Generally $a_{N+k} < c^k a_N$. Now $\sum_k c^k a_N$ converges (geometric series), so by the Basic Comparison Test, $\sum_k a_{N+k}$ converges. So $\sum_k a_k$ converges.

  A similar argument does the case $r > c > 1$. By the liminf variant of (6) in Homework 1 there exists N with $\frac{a_{k+1}}{a_k} > c$ for $k \geq N$. In this case $a_{N+k} > c^k a_N$. Now $c^k a_N \to \infty$ as $k \to \infty$, so by the Divergence test, $\sum_k a_{N+k}$ diverges. So $\sum_k a_k$ diverges.]

- From Homework 3 Question 3 (a) below it is easy to see that the root test is more powerful theoretically than the ratio test. That is if the ratio test works to prove convergence or divergence, then the root test would give the same conclusion. However the converse is not true (if the root test works, the ration test might not). See Homework 3 Question 2.

- **Condensation test** Suppose that $a_0 \geq a_1 \geq a_2 \geq \cdots \geq 0$, and that $\lim_k a_k = 0$. Then $\sum_k a_k$ converges iff $\sum_k 2^k a_{2^k}$ converges.

  [Proof: Let $s_n = \sum_{k=0}^n a_k$ and $t_n = \sum_{k=0}^n 2^k a_{2^k}$. If $n < 2^k$ then

$$\begin{aligned} s_n &\leq a_1 + (a_2 + a_3) + (a_4 + \cdots + a_7) + \cdots + (a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} = t_k. \end{aligned}$$

  Thus if (t_k) is bounded then (s_n) is bounded; and so by the fact in the paragraph before the harmonic series a few pages back, $\sum_k a_k$ converges if $\sum_k 2^k a_{2^k}$ converges. If $n > 2^k$ then

$$\begin{aligned} s_n &\geq a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\geq \frac{1}{2} a_1 + a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k} = \frac{1}{2} t_k. \end{aligned}$$

  So $t_k \leq 2s_n$. Thus if (s_n) is bounded then (t_k) is bounded; and so by the fact in the paragraph before the harmonic series a few pages back, $\sum_k 2^k a_{2^k}$ converges if $\sum_k a_k$ converges.

- As an application of the condensation test note that the harmonic series $\sum_{n=1}^\infty \frac{1}{n}$ diverges because $\sum_k 2^k \frac{1}{2^k} = \sum_k 1 = \infty$.

Homework 3 (due Thursday January 29).

(1) Test for convergence (giving reasons): (a) $\sum_{n=1}^\infty \frac{n^3}{2^n}$, (b) $\sum_{n=2}^\infty \frac{1}{(\log n)^2}$, (c) $\sum_{n=1}^\infty n^{-1-\frac{1}{n}}$, (d) $\sum_{n=2}^\infty \frac{1}{(\log n)^{\log n}}$, (e) $\sum_{n=1}^\infty (\sqrt[n]{n} - 1)^n$, (f) $\sum_{n=1}^\infty \frac{\sqrt{1+n^2}-n}{\sqrt{n}}$.

(2) Suppose $\sum_{n=1}^\infty a_n$ is the series $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \cdots$. Find $\limsup_n \frac{a_{n+1}}{a_n}$, $\liminf_n \frac{a_{n+1}}{a_n}$ and $\limsup_n a_n^{\frac{1}{n}}$. Can we conclude that $\sum_n a_n$ converges using the ratio test? Using the root test?

(3) (a*) Show (or look up) that

$$\liminf_n |s_{n+1}/s_n| \leq \liminf_n |s_n|^{1/n} \leq \limsup_n |s_n|^{1/n} \leq \limsup_n |s_{n+1}/s_n|,$$

for any sequence $\{s_n\}$ in $\mathbb{R} \setminus \{0\}$.

(b) From (a) it is easy to see that the root test is more powerful theoretically than the ratio test. For example, deduce that if $\{|s_{n+1}/s_n|\}$ converges then $\{|s_n|^{1/n}\}$ converges to the same limit.

(c) Calculate $\lim_{n\to\infty} (n!)^{1/n}$.

4. Absolute and conditional convergence

A series $\sum_k a_k$ is called *absolutely convergent* if $\sum_k |a_k|$ converges. Recall that a_k here could be a complex number (you can view complex numbers as elements of $\mathbb{R}^2$ here).

- Any absolutely convergent series is convergent.

  [Proof: By the Cauchy test above, since $\sum_k |a_k|$ converges, given $\epsilon > 0$ there exists an $N \geq 0$ such that

$$|a_{n+1} + a_{n+2} + \cdots + a_m| \leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_m| < \epsilon, \qquad m > n \geq N.$$

  By the Cauchy test again, $\sum_k a_k$ converges.]

- The converse is false, a series may be convergent, but not absolutely convergent. Such a series is called *conditionally convergent.*

- **The Alternating Series Test (a.k.a. Leibniz Test)/Alternating Series approximation:** Suppose that $a_0 \geq a_1 \geq a_2 \geq \cdots$, and that $\lim_k a_k = 0$. Then $a_0 - a_1 + a_2 - a_3 + \cdots$ (which in sigma notation is $\sum_{k=0}^{\infty} (-1)^k a_k$) converges, and moreover $|s_n - \sum_{k=0}^{\infty} (-1)^k a_k| \leq a_n$ for all n, where s_n is the nth partial sum $\sum_{k=0}^{n-1} (-1)^k a_k$.

  [Proof: We have noticed before that if $m > n$ then $s_m - s_n = b_n + b_{n+1} + \cdots + b_{m-1}$, where b_k is the kth term. Here $b_k = (-1)^k a_k$, where a_k is as above. Note that $a_{n+k} - a_{n+k+1} \geq 0$, so $a_n - a_{n+1} + a_{n+2} - a_{n+3} + \cdots \geq 0$. Hence

$$|s_m - s_n| = a_n - a_{n+1} + a_{n+2} - \cdots = a_n - (a_{n+1} - a_{n+2}) - (a_{n+3} - a_{n+4}) - \cdots \leq a_n,$$

  since $a_{n+k+1} \leq a_{n+k}$. It follows that (s_n) is Cauchy, so convergent. That is, $\sum_{k=0}^{\infty} (-1)^k a_k$ converges. Letting $m \to \infty$ and using a fact about sequences from the prerequisite, we have $|\sum_{k=0}^{\infty} (-1)^k a_k - s_n| \leq a_n$.]

- Example. Approximate the sum of $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4}$ with an error of less than 0.001.

  Solution. The error in using s_n to approximate the sum is $< |a_{n+1}| = \frac{1}{(n+1)^4}$ (note the starting index of this series is 1, not 0, which means we have to change the formula in the last result slightly). Now $\frac{1}{(n+1)^4} < 0.001$ if $(n+1)^4 > 1000$. Choosing $n = 5$ will work. So an approximation with an error of less than 0.001 is $s_5 = \sum_{k=1}^{5} \frac{(-1)^{k+1}}{k^4} = 0.94754$ (calculator).

- **Exercise.** Test for convergence/absolute or conditional convergence: $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n!}$.

- If $\sum_{k=1}^{\infty} a_k$ converges absolutely then $|\sum_{k=1}^{\infty} a_k| \leq \sum_{k=1}^{\infty} |a_k|$.

  [Proof. Take the limit as $n \to \infty$ in the inequality, nd using a fact about sequences from the prerequisite, we have $|\sum_{k=1}^{n} a_k| \leq \sum_{k=1}^{n} |a_k|$.]

- If $\sum_{k=1}^{\infty} a_k$ is a series, and $f : \mathbb{N} \to \mathbb{N}$ is a bijection, then the series $\sum_{k=1}^{\infty} a_{f(k)}$ is called a 'rearrangement' of $\sum_{k=1}^{\infty} a_k$. It is not hard to see that a rearrangement of a convergent series need not converge.

  Theorem Any 'rearrangement' of an absolutely convergent series is convergent and has the same sum.

  [Proof: Suppose that $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, and that $\epsilon > 0$ is given. Then $\sum_{k=1}^{\infty} a_k$ is convergent too, with sum s say, so $s_n \to s$ where $s_n = \sum_{k=1}^{n} a_k$. Choose N with $\sum_{k=N}^{\infty} |a_k| < \frac{\epsilon}{2}$ (see HW 2 Q 1), and $|s_n - s| < \frac{\epsilon}{2}$ for $n \geq N$. Choose $M \in \mathbb{N}$ such that $\{1, 2, \cdots, N\} \subset \{f(1), f(2), \cdots, f(M)\}$ (this is possible since f is bijective–why?). Let $t_n = \sum_{k=1}^{n} a_{f(k)}$. If $n > M$ we have that all the terms a_k in $s_n = \sum_{k=1}^{n} a_k$ with $k \leq N$ cancel with terms $a_{f(j)}$ in $t_n = \sum_{k=1}^{n} a_{f(k)}$. So (and also using the triangle inequality),

  $$|s_n - t_n| \leq |a_{N+1}| + |a_{N+2}| + \cdots < \frac{\epsilon}{2}.$$

  So

  $$|t_n - s| \leq |t_n - s_n| + |s_n - s| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \qquad n \geq M.$$

  Hence $t_n \to s$. So the rearrangement is convergent and has the same sum.]

- Dirichlet test: Let $\sum_k a_k$ be a series whose partial sums form a bounded sequence. Suppose (b_n) is a decreasing sequence with limit 0. Then $\sum_k a_k b_k$ converges.
- Abel's test: Suppose that $\sum_k a_k$ converges and b_n is a monotonic convergent sequence. Then $\sum_k a_k b_k$ converges.

We will not prove the last two results (see e.g. Apostol's text for proofs, they are not hard).

Homework 4 (due Tuesday February 3).

(1) Let $\sum_{k=1}^{\infty} a_k$ be a series, and suppose that $1 \leq n_1 < n_2 < \cdots$ are integers. Let $b_1 = \sum_{k=1}^{n_1} a_k, b_2 = \sum_{k=n_1+1}^{n_2} a_k, b_3 = \sum_{k=n_2+1}^{n_3} a_k, \cdots$. We call $\sum_{k=1}^{\infty} b_k$ a series obtained from $\sum_{k=1}^{\infty} a_k$ *by adding parentheses.* Prove that if $\sum_{k=1}^{\infty} a_k$ converges then $\sum_{k=1}^{\infty} b_k$ converges and has the same sum.

Also: (a) If $\sum_{k=1}^\infty a_k$ is a nonnegative series, prove that $\sum_{k=1}^\infty a_k$ converges iff $\sum_{k=1}^\infty b_k$ converges. (b)* If the sequence $(n_{k+1} - n_k)$ is bounded and $\lim_n a_n = 0$, show that $\sum_{k=1}^\infty a_k$ converges iff $\sum_{k=1}^\infty b_k$ converges.

(2) Prove the 'fundamental fact about power series' in the last 'bullet' before Homework 1.

(3) Suppose that $z_n = a_n + ib_n$, where $a_n,\ b_n \in \mathbb{R}$. Show that (a) $\sum z_n$ is convergent if and only if both $\sum a_n$ and $\sum b_n$ are convergent; (b) $\sum z_n$ is absolutely convergent if and only if both $\sum a_n$ and $\sum b_n$ are absolutely convergent; (c) $\sum_{n=1}^\infty \left(\frac{(-1)^n}{\sqrt{n}} + \frac{i}{n^2}\right)$ is convergent but not absolutely convergent.

(4) If $\sum_n a_n$ converges absolutely, show that $\sum_n a_n^2$ and $\sum_n \frac{a_n}{1+a_n}$ converge absolutely (you may assume if you wish that that no $a_n = -1$). If $\sum_n a_n$ diverges show $\sum_n n\, a_n$ diverges.

(5)* Prove or look up a proof of Riemann's result that any conditionally convergent series has the property that if $a \in \mathbb{R}$ is given, there is a rearrangement of the series which has sum a. Show also that there is a rearrangement of the series which diverges.

5. Double sums

This section is really about interchanging double sums: that is when $\sum_{n=1}^\infty \sum_{m=1}^\infty$ equals $\sum_{m=1}^\infty \sum_{n=1}^\infty$.

If we have numbers $a_{m,n} \geq 0$ for all $m, n \in \mathbb{N}$, define $\sum_{n,m=1}^\infty a_{m,n}$ to be the supremum over all 'partial sums' $S_N = \sum_{n,m=1}^N a_{m,n}$, for $N \in \mathbb{N}$.

Theorem 5.1. *If $a_{m,n} \geq 0$ for all $m, n \in \mathbb{N}$, then the following sums are equal:*

$$\sum_{n=1}^\infty \left(\sum_{m=1}^\infty a_{m,n}\right) = \sum_{m=1}^\infty \left(\sum_{n=1}^\infty a_{m,n}\right) = \sum_{n,m=1}^\infty a_{m,n}.$$

Proof. We leave this an exercise in sups of positive numbers, using the fact that the three double sums here are really just

$$\sup\{\sum_{n=1}^N \sup\{\sum_{m=1}^M a_{m,n} : M \in \mathbb{N}\} : N \in \mathbb{N}\},$$

$$\sup\{\sum_{m=1}^M \sup\{\sum_{n=1}^N a_{m,n} : N \in \mathbb{N}\} : M \in \mathbb{N}\},$$

and $\sup\{\sum_{n=1}^N \sum_{m=1}^N a_{m,n} : N \in \mathbb{N}\}$. □

Next we allow the $a_{m,n}$ to be negative as well as positive; or even complex. Again we have 'partial sums' $S_N = \sum_{n,m=1}^N a_{m,n}$, for $N \in \mathbb{N}$. Now we say that

$\sum_{n,m} a_{m,n}$ converges if there is a number s such that for all $\epsilon > 0$ there exists $K \geq 1$ such that

$$|s - \sum_{n=1}^{N} \sum_{m=1}^{M} a_{m,n}| < \epsilon$$

whenever $N \geq K$, and $M \geq K$. We write this number s as $\sum_{n,m=1}^{\infty} a_{m,n}$, or sometimes simply as $\sum_{n,m} a_{m,n}$.

We remark that if $a_{m,n} \geq 0$ for all $m, n \in \mathbb{N}$ then saying that $\sum_{n,m} a_{m,n}$ converges is the same as saying that $\sum_{n,m=1}^{\infty} a_{m,n}$ (in the sense defined before Theorem 5.1) is finite. See Exercises below.

Theorem 5.2. *If $a_{m,n}$ are real (or complex) for all $m, n \in \mathbb{N}$, and if $\sum_{n,m=1}^{\infty} |a_{m,n}| < \infty$, then $\sum_{n,m} a_{m,n}$ converges, and all of the following sums are finite and we have:*

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_{m,n} \right) = \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{m,n} \right) = \sum_{n,m=1}^{\infty} a_{m,n} = \lim_{N \to \infty} S_N.$$

Proof. We leave this as an exercise for the graduate students in the class (note that the last '=' is almost immediate by taking $N = M$ in the definition of the sum $\sum_{n,m=1}^{\infty} a_{m,n}$). □

Double series obey the same rules as ordinary series. For example,

$$\sum_{n,m=1}^{\infty} a_{m,n} + \sum_{n,m=1}^{\infty} b_{m,n} = \sum_{n,m=1}^{\infty} (a_{m,n} + b_{m,n}), \qquad \sum_{n,m=1}^{\infty} c a_{m,n} = c \sum_{n,m=1}^{\infty} a_{m,n},$$

provided the first two double series converge, and c is a constant (scalar). The proofs are the same as before.

We can write a convergent double series $\sum_{n,m=1}^{\infty} a_{m,n}$ as an ordinary series. Indeed if $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a bijective function set $b_k = a_{g(k)}$ and consider $\sum_k b_k$.

Theorem 5.3. *If $a_{m,n}$ are real (or complex) for all $m, n \in \mathbb{N}$, and if $\sum_{n,m=1}^{\infty} |a_{m,n}| < \infty$, and b_k is as defined above, then $\sum_k b_k$ converges (absolutely), and equals $\sum_{n,m=1}^{\infty} a_{m,n}$.*

Proof. We can think of $S_N = \sum_{n,m=1}^{N} a_{m,n}$ as the N-th partial sum of the series $\sum_k c_k$ where $c_1 = a_{1,1}$, and

$$c_2 = a_{1,2} + a_{2,2} + a_{2,1}, \quad c_3 = a_{1,3} + a_{2,3} + a_{3,3} + a_{3,1} + a_{3,2}, \quad \cdots .$$

The associated series

$$a_{1,1} + a_{1,2} + a_{2,2} + a_{2,1} + a_{1,3} + a_{2,3} + a_{3,3} + a_{3,1} + a_{3,2} + a_{1,4} + \cdots$$

(note inserting parentheses in this series gives $\sum_k c_k$) converges absolutely since

$$|a_{1,1}| + |a_{1,2}| + |a_{2,2}| + |a_{2,1}| + |a_{1,3}| + \cdots$$

has partial sums dominated by $\sum_{n,m=1}^{\infty} |a_{m,n}| < \infty$. So by Homework 4 Question 1, $\sum_k c_k$ converges. Moreover $\sum_k b_k$ is a rearrangement of the series in the second last centered formula. So by our earlier theorem on rearrangement of series, $\sum_k b_k$ converges (absolutely), and by Homework 4 Question 1 its sum equals $\sum_k c_k = \lim_N S_N = \sum_{n,m=1}^{\infty} a_{m,n}$. We have also used here the fact about S_N at the start of the proof. The last '=' uses a formula from the previous theorem. □

The *Cauchy product* of two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is the series $\sum_{n=0}^{\infty} c_n$ where

$$c_n = \sum_{k=0}^{n} a_k \, b_{n-k}.$$

A theorem of Mertens (see Apostol) says that if $\sum_{n=0}^{\infty} a_n$ converges absolutely with sum s, and if $\sum_{n=0}^{\infty} b_n$ converges with sum t, then the Cauchy product series converges and has sum st. A special case of this is found in Homework 5 Question 6 below.

Homework 5 (due Thursday February 5).

(1) Discuss the convergence of $\sum_{n,m} \frac{nm}{n^2+m^2}$ and $\sum_{n,m} 2^{-(n^2+m^2)}$.

(2) Prove Theorem 5.1 in detail.

(3) If $a_{m,n} \geq 0$ for all $m, n \in \mathbb{N}$ show that $\sum_{n,m} a_{m,n}$ converges iff $\sum_{n,m=1}^{\infty} a_{m,n}$ (in the sense defined before Theorem 5.1) is finite.

(4)* Prove Theorem 5.2.

(5) If $\sum_n a_n$ and $\sum_n b_n$ are absolutely convergent series with sums s and t respectively, show that $\sum_{n,m} a_n \, b_m$ is an absolutely convergent double series whose sum is st.

(6) If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge absolutely, show that the Cauchy product series converges (absolutely) and has sum st. [Hint: This follows from Homework 5 Question 5 and Theorem 5.3, and Homework 4 Question 1, because the Cauchy product series is one of the kind considered in Theorem 5.3, but then with parentheses added as in Homework 4 Question 1.]

(7) If s_n is the nth partial sum of a series $\sum_k a_k$, define the *Cesáro means* to be the sequence (σ_n) defined by $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. We say that $\sum_k a_k$ is *Cesáro summable* if the sequence (σ_n) converges, and in this case $\lim_n \sigma_n$ is called the *Cesáro sum* of the series $\sum_k a_k$. Prove that if $\sum_k a_k$ converges with sum s then it is Cesáro summable and its Cesáro sum is s.

END OF CHAPTER 0