MATH 4389–SOME FUNDAMENTAL FACTS ABOUT POWER SERIES

A power series is an expression of the form

$$\sum_{k=0}^{\infty} c_k x^k.$$

Here x and the c_k can be complex numbers if you wish, but in the discussion below we assume they are real. A **power series centered at** c is an expression of form $\sum_{k=0}^{+\infty} a_k(x-c)^k$. Any power series centered at c can be turned into a power series of the first type (centered at 0) by letting $u = x - c$. So we only consider power series $\sum_{k=0}^{\infty} c_k x^k$ below. The results for power series centered at c will be analogous, but for example the interval of convergence will be centered at c.

- Given a power series as above we set $A = \lim_{n\to+\infty} \sqrt[n]{|c_n|} = \lim_{n\to+\infty} |c_n|^{\frac{1}{n}}$. (For the sophisticated, if this limit does not exist set $A = \limsup_{n\to+\infty} \sqrt[n]{|c_n|}$.) Set $A = +\infty$ if these nth roots are unbounded. We set

$$R = \begin{cases} +\infty, & A = 0, \\ \frac{1}{A}, & 0 < A < +\infty, \\ 0, & A = +\infty \end{cases}$$

  and we call R the **radius of convergence of the power series**. The 'interval of convergence' is the set of numbers x for which the series $\sum_{k=0}^{\infty} c_k\, x^k$ converges.
- If $R = 0$ the power series only converges at 0, and then the interval of convergence is $\{0\}$. This is the 'trivial case'. So in all of the following items suppose that $R > 0$.
- If $R > 0$ then the power series converges (absolutely) if $|x| < R$ and diverges if $|x| > R$. [Picture drawn in class.] Thus the interval of convergence consists of the interval from $-R$ to R, with the possible (but not necessary) inclusion of one or more of the endpoints.
- Another formula for the number R above is $\lim_{n\to+\infty} \frac{|c_{n+1}|}{|c_n|}$. This sometimes does not exist, but when it does it equals $\lim_{n\to+\infty} |c_n|^{\frac{1}{n}}$.
- If a power series converges for a fixed real number d, and diverges at $-d$, then the radius of convergence is $|d|$ (so $R = |d|$), and the interval of convergence consists of the interval from $-d$ to d with d included but $-d$ excluded.

- The sum of the power series is a continuous function on its interval of convergence. Call it the 'sum function' and write it as $f(x)$, for x in the interval of convergence. In fact f is differentiable, and indeed infinitely many times differentiable, on $(-R, R)$. Also $f'(x) = \sum_{k=1}^{\infty} kc_k\, x^{k-1}$ for $|x| < R$ (the latter series converges here, with sum $f(x)$). Similarly $f''(x) = \sum_{k=2}^{\infty} k(k-1)c_k\, x^{k-2}$, and so on. All these new power series have the same radius of convergence R.
- It follows by setting $x = 0$ that $c_n = \frac{f^n(x_0)}{n!}$, for all $n = 0, 1, 2, \cdots$.
- Abel's theorem: suppose that a power series converges at one of the endpoints d of $(-R, R)$ (so d is R or $-R$ in this case). Let $f(x)$ be the sum function (set $f(d) = \infty$ if the power series converges to ∞ at d). Then $f(d)$ equals the one-sided limit of $f(x)$ as x approaches this endpoint. Moreover the power series converges uniformly to $f(x)$ on the compact interval with endpoints 0 and d.
- Do not confuse power series and Taylor series. This page is not about Taylor series. You can read up on e.g. wikipedia about those. However, if $R > 0$ then the Taylor series of the 'sum function' $f(x)$ above is $\sum_{k=0}^{\infty} c_k\, x^k$. That is, the sum function $f(x)$ of a power series on its interval of convergence, has Taylor series equal to the original power series.
- Sums, products, equality, etc, of two power series (omitted).
- The integrated power series of $\sum_{k=0}^{\infty} c_k\, x^k$ is $\sum_{k=0}^{\infty} \frac{c_k}{k+1}\, x^{k+1}$. This has the same radius of convergence R.
- If $f(x)$ is the sum function of $\sum_{k=0}^{\infty} c_k\, x^k$, and if $F(x)$ is the sum function of the integrated power series, then $\int_0^x f(t)\, dt = F(x)$, for $|x| < R$.