Department of Mathematics, University of Houston
Math 6342–Topology. David Blecher
Homework 2

Instructions. Some of these problems are difficult, others are easy. You may ignore problems you have done before in previous classes (which you remember how to do), or which are 'starred'. You are encouraged to work with others, form study groups, and so on; however do not copy homework you are required to turn in.

1. Describe the topologies on a three point set. Also say (roughly) which of these topologies are comparable (i.e. finer or coarser).

2. Which of the topologies on the set in Exercise 1 are Hausdorff? Which are metrizable? (Note that every metrizable topology is Hausdorff.)

3. Let X be a set, and $\mathcal{T}_c = \{U \subset X : X - U \text{is countable or all of } X\}$, and let $\mathcal{T}_\infty = \{U \subset X : X - U \text{ is infinite or empty or all of } X\}$. Are these topologies on X?

4. (1) If $\{\tau_\alpha\}$ is a family of topologies on X, show that $\bigcap \tau_\alpha$ is a topology on X.
   (2) Let $\{\tau_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology containing all the collections τ_α, and a unique largest topology contained in all τ_α.
   (3) If $X = \{a, b, c\}$, let
   $$\tau_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \qquad \text{and} \qquad \tau_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$
   Find the smallest topology containing τ_1 and τ_2, and the largest topology contained in τ_1 and τ_2.

5. Describe the closure of the set $\{1\}$ in the topology on $\mathbb{R}$ with subbasis the intervals (a, ∞) for $a \in \mathbb{R}$.

6. The *Sorgenfrey line* is $\mathbb{R}$ with topology with subbasis the intervals $[a, b)$ for $a < b$ in $\mathbb{R}$. Show that this is strictly finer that the standard topology. From your knowledge of 'undergrad topology', what topological properties do you think it has? (e.g. compact, connected, path connected, separable, metrizable, Hausdorff? We will revisit this question later when we know more–for example you may not know what some of these words mean, and we may not know enough at this point to test if some of these are true).

7. If $(X, \leq)$ is a totally ordered set, define the order topology to be the topology with subbasis the sets $\{x \in X : x < y\}$ and $\{x \in X : y < x\}$, for $y \in X$. If $I = [0, 1]$ with its usual ordering, define the dictionary ordering on $I \times I$ to be: $(s, t) \leq (a, b)$ if $s < a$ or $s = a$ and $t \leq b$. Check that this is a total ordering. We call $I \times I$ with the induced order topology the *ordered square*. Compare this to the standard topology on $I \times I$; for example is it finer or coarser; and from your knowledge of 'undergrad topology', what topological properties do you think it has? (e.g. compact, connected, path connected, separable, metrizable, Hausdorff? We will revisit this question later when we know more–for example you may not know what some of these words mean, and we may not know enough at this point to test if some of these are true).

8. Show that the countable collection

$$\mathcal{B} = \{(a, b) : a < b, a \text{ and } b \text{ rational}\}$$

is a basis for the standard topology on $\mathbb{R}$.