Department of Mathematics, University of Houston
Math 6342. Topology. David Blecher
Homework 4

Instructions. Some of these problems are difficult, others are easy. You are encouraged to work with others, form study groups, and so on; however do not copy homework unless you've tried hard to do the problem, and even then you need to understand the solution and write it in your own words. Never copy homework you are required to turn in.

1. Let (X, τ) be a topological space with the property that every convergent net in X has a unique limit. Show that (X, τ) is Hausdorff.

2. (a) Give a test for a set to be open, in terms of nets.
   (b) Suppose X is a set with two topologies σ and τ. Suppose that every net $\{x_i\}$ in X which converges to a point $x \in X$ with respect to the topology σ, also converges to x with respect to the topology τ. Show that $\tau \subset \sigma$. Show that $\tau = \sigma$ if the converse is also true, namely that every net $\{x_i\}$ which converges to a point $x \in X$ with respect to the topology τ, also converges to x with respect to the topology σ.
   (c) Think about this a moment and realize that this says that the convergence of nets completely determines the topology.

3. Let (X, τ) be a topological space, and suppose that Y is a subset of X, endowed with the relative topology τ_Y. Prove that a net in Y is convergent to a point in Y with respect to the topology τ if and only if it is convergent to that point with respect to the relative topology τ_Y.

4. Let (X, τ) be a topological space, and fix $x \in X$. Show that a net $(x_\lambda)_{\lambda \in \Lambda}$ in X converges to x, if and only if every subnet of (x_λ) possesses a subnet which converges to x.

5. Concerning nets of real or complex numbers:
   (a) Prove a few of the properties stated in 1.3.7 about nets of real or complex numbers.
   (b) Show that a convergent net of real numbers need not be bounded; but is *eventually bounded*: that is there is a constant $M \geq 0$, and $\lambda_0 \in \Lambda$, such that $|x_\lambda| \leq M$ for all $\lambda \geq \lambda_0$.
   (c) If you have time, prove a few of the properties stated in 1.3.9 about series of numbers.

6. A Cauchy net in a metric space (X, d) is a net (x_t) in X such that for every $\epsilon > 0$ there exists an t_0 such that $d(x_s, x_t) < \epsilon$ whenever $s, t \geq t_0$. Prove that in a complete metric space, every Cauchy net converges. [Hint: Choose t_n with $d(x_s, x_t) < 2^{-n}$ whenever $s, t \geq t_n$. We can ensure that (t_n) is increasing. Consider (x_{t_n}). Recall what happens if a Cauchy sequence has a convergent subsequence.]