Department of Mathematics, University of Houston
Math 6342. Topology. David Blecher. Homework 5

Instructions. **Some of these problems are difficult, others are easy. You may ignore problems you have done before in previous classes (which you remember how to do). You are encouraged to work with others, form study groups, and so on; however do not copy homework you are required to turn in.**

1. Prove that $\mathbb{R}$ is homeomorphic to $(0, 1)$ (put in all details). Also, find a continuous function $f : \mathbb{R} \to \mathbb{R}$ which is not open. [Hint: one may use common functions from Calculus in each part.]

2. It is said that a 'topologist' is a mathematician who cant see the difference between a donut and a coffee cup. How about a teapot? [Hint: some kind of donut. We can ignore the lid, which is separate and easy. You need not give a proof, maybe just a convincing series of pictures.]

3. Which of the topologies on the 3 point set found in HW 2, are homeomorphic?

4. Show that a function $f : X \to Y$ between metric spaces is continuous if and only if whenever a sequence (x_n) converges in X to a point $x \in X$, then $f(x_n) \to f(x)$.

5. Show that if f is a continuous function between topological spaces, then $f(\bar{A}) \subset \overline{f(A)}$ for a subset A.

6. Prove Theorem 1.4.5 (d).

7. Identify $\mathbb{R}/\mathbb{Q}$, the quotient topological space obtained from $\mathbb{R}$ by identifying two numbers if they differ by a rational number. [Hint: if U is open in $\mathbb{R}/\mathbb{Q}$ then $q^{-1}(U)$ is open and saturated. What are the open saturated sets in $\mathbb{R}$?]

8. Show that if $f : X \to Y$ is continuous and has a continuous right inverse (see HW1), then f is a quotient map. [A commonly met case of this is when we have a *retraction* of Y onto a subspace X of Y (here $f : X \to Y$ is the inclusion map, and we are saying that there is a continuous $g : Y \to X$ with $g(x) = x$ for all $x \in X$).]

9. Let $\pi_1 : R \times R \to R$ be the projection of the plane onto its first coordinate. Let $A = \{(x, y) : x \geq 0 \text{ or } y = 0\}$, and let $q = (\pi_1)_{|A}$. Prove that q is a quotient map that is neither open nor closed. [Hint: use Q 8 for part of this.]

10. Prove that the quotient of the triangle given by the labelling scheme aba^{-1} is a closed disk. Prove that the quotient of the square given by the labelling scheme $abac$, is homeomorphic to the Mobiüs strip. Prove that the two descriptions in 1.5.12 of the projective plane (one of them as the quotient of a sphere, the other from the labelling scheme $abab$) give homeomorphic spaces.

11. Prove the exercises left at the end of Section 1.6.

12. Prove that $(X_1 \times \cdots X_{n-1}) \times X_n$ is homeomorphic to $X_1 \times \cdots \times X_n$.