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Math 6342. Topology. David Blecher
Homework 6 Compactness

Instructions. **Some of these problems are difficult, others are easy. You may ignore problems you have done before in previous classes (which you remember how to do). You are encouraged to work with others, form study groups, and so on; however do not copy homework you are required to turn in.**

0. Do the exercises before 2.1.4 and 2.1.10.

1. Is $[0, 1]$ compact in the topology $\mathcal{T}_c$ on R in HW 2 Q 3? [Hint: think about what an open set looks like in that topology.]

2. If A, B are disjoint compact subsets of a Hausdorff space X, show that there are two disjoint open sets one containing A and the other containing B. [Hint: Similar to 2.2.2/2.1.5 in the notes.]

3. Let $f : X \to Y$ be a continuous function between topological spaces, and consider the graph of f:
$$\mathcal{G}(f) \ = \ \{(x, y) \in X \times Y \ : \ y = f(x)\} \ .$$
Show that if Y is Hausdorff then $\mathcal{G}(f)$ is closed with respect to the product topology. Show that the converse is true if Y is also compact, that is, if $\mathcal{G}(f)$ is closed then f is continuous. [Hint: That $\mathcal{G}(f)$ is closed if f is continuous using e.g. the criterion in terms of nets for a set to be closed, and what it means for a net in $X \times Y$ to converge. The other direction is harder; one could use e.g. the criterion in terms of nets for continuity of f, Q 4 below, and the fact that accumulation points are limits of subnets.]

4. Let (x_λ) be a net in a compact space X, and $x \in X$. Show that $x_\lambda \to x$ if and only if every convergent subnet converges to x. Deduce that if X is also Hausdorff then a net in X is convergent if and only if it has one and only one accumulation point. [Hint: One could use e.g. the matching question in Homework 4.]

5. (a) Let C be a compact subset of a metric space (X, d). Show that for every open covering σ of C, there is a number $\epsilon > 0$ such that for each $x \in C$, the ball center x of radius ϵ is contained in some element from σ. [This is sometimes called a Lebesgue number, and the result *the Lebesgue covering lemma*. This question could properly belong in our senior analysis sequence, or in one of our 6000 level analysis classes.]
   (b) As an application of this, show that a continuous function betweem metric spaces, whose domain is compact, is uniformly continuous.

6. Show that the union of two compact sets is compact. Show that the intersection of arbitrarily many closed sets, one of which is compact, is compact.

7. Prove that the quotient $X/\sim$ of a compact Hausdorff space is Hausdorff iff $\sim$ is a *closed relation*, that is $\{(x, y) \in X \times X : x \sim y\}$ is closed in the product topology (Hint: for the easier direction you could use nets: if $(x_t, y_t) \to (x, y)$ and $x_t \sim y_t$ show $x \sim y$. For the

harder direction you could use Homework 7 question 8a, which says that if $p : X \to Y$ is a continuous closed surjective map with $p^{-1}(\{y\})$ compact for every $y \in Y$, and if X is Hausdorff then so is Y. Note that if $X/\sim$ is Hausdorff then the canonical map $q : X \to X/\sim$ is closed by 2.1.9. If you get stuck, then prove the easier version of this question in which we also assume that the canonical map $q : X \to X/\sim$ is open).

8. Let C be the usual Cantor set, obtained from $[0, 1]$ by the usual repeated 'removing the open middle third subinterval' construction (e.g. Munkres p. 178). Show that C is compact. Show that C has no isolated points [Hint: Given $\epsilon > 0$ choose $3^{-n} < \epsilon$, and note that every point in the set A_n from the nth stage of the construction of the Cantor set is within distance $\leq 3^{-n}$ of another point in the boundary of A_n. Note that this question and 9 a–d usually appear in a graduate Real Variables course too.] What is the cardinality of C?

9. Let C be the Cantor set constructed in the last question.
   (a) Show that C has empty interior, and is 'zero dimensional' (that is, it has a countable basis of clopen sets), and 'totally disconnected' (in that the only connected subsets are singletons). [Hint: consider the size of the subintervals in the set A_n above; these subintervals are the desired basis.]
   (b) Show that C is the set of real numbers of the form $\sum_{n=1}^{\infty} \alpha_n 3^{-n}$, where $\alpha_n = 0$ or 2. Indeed show that every $x \in C$ has a unique such series (that is, the α_n above are uniquely determined by x.) [Hint: This is just the 'base 3' expansion of a number, and connecting that with the set A_n above.] Which such series correspond to endpoints of the subintervals in the sets A_n above?
   (c) Show that C is homeomorphic to $\{0, 1\}^{\mathbb{N}}$, the latter with the product topology. [Hint: use (b).]
   (d) Define a function $f : C \to [0, 1]$ by $f(x) = \sum_{n=1}^{\infty} \alpha_n 2^{-n-1}$, when $x = \sum_{n=1}^{\infty} \alpha_n 3^{-n}$ as in (b). Show that f is well defined, continuous, and surjective.
   (e) Suppose that $x, y, z \in C$ with $y < x < z$, and $x = \frac{y+z}{2}$. Write $x = \sum \alpha_n 3^{-n}, y = \sum \beta_n 3^{-n}, z = \sum \gamma_n 3^{-n}$, as in (b). Let m be the largest integer such that $\alpha_n = \beta_n = \gamma_n$ for all $n < m$. Show that one of the following two cases occurs: either $\alpha_m = \beta_m = 0, \gamma_m = 2$ in which case $\alpha_n = 2$ and $\beta_n = \gamma_n = 0$ for all $n > m$; or $\alpha_m = \gamma_m = 2, \beta_m = 0$ in which case $\alpha_n = 0$ and $\beta_n = \gamma_n = 2$ for all $n > m$. Deduce from this that $R \setminus C$ contains an interval, one of whose endpoints is x (which in particular shows that C is what is called 'totally disconnected').
   (f) Show that for each closed subset F of C, there is a continuous function $g : C \to F$ such that $g(x) = x$ for all $x \in F$. [Hint: Define $g(x)$ to be the nearest point to x in F. If there are two such nearest points y and z, then we may suppose that $y < x < z$, and $x = \frac{y+z}{2}$. In this case define $g(x) = y$ if $(x, x + \epsilon) \subset R \setminus C$, or $g(x) = z$ if $(x - \epsilon, x) \subset R \setminus C$ (using (e)).
   (g*) Show (using (f)) that for every compact metric space X there is a continuous surjective function from C onto X. That is, X is a *continuous image* of C, and hence X is homeomorphic to a quotient of C by an equivalence relation. [Hint: Identify C with $\{0, 1\}^{\mathbb{N}}$ as in (c). One way to proceed is via the use of 2.3.3. A part of the above proof essentially shows that every compact Hausdorff space is a continuous image of a closed subset of some power of $\{0, 1\}$. We remark that one can show that any two nonempty compact totally disconnected (or zero-dimensional) metrizable spaces without isolated

points are homeomorphic (due to Brouwer, and related to Stones theorem for Boolean algebras).

10. Say that a function $f : X \to Y$ between topological spaces is *proper* if $f^{-1}(K)$ is compact for every compact set K in Y. Show that if $f(C)$ is closed in Y for all closed C in X, and if $f^{-1}(\{y\})$ is compact for every $y \in Y$, then f is proper. [Hint: you could use the fact that $\forall y \in Y$ and open U in X containing $f^{-1}(\{y\}), \exists$ open W containing y with $f^{-1}(\{y\}) \subset f^{-1}(W) \subset U$. Indeed $W = f(U^c)^c$ does the job here.]

11. Show that a nonempty compact Hausdorff space with no isolated points is uncountable.

12. Suppose that A is a subspace of a topological space X, and suppose that K is a compact topological space. If U is an open set in $X \times K$ containing $A \times K$, prove that there is an open set V in X containing A such that $V \times K \subset U$. This is sometimes called the *tube lemma*.