Department of Mathematics, University of Houston
Math 6342. Topology. David Blecher, Fall 2003
Homework 7

Instructions. Some of these problems are difficult, others are easy. You may ignore problems you have done before in previous classes (which you remember how to do). You are encouraged to work with others, form study groups, and so on; however do not copy homework you are required to turn in.

1. (The Sorgenfrey line) This question will not be graded. Let R be the real number line with basis the set of (finite) half open intervals $[a, b)$ (recall that with this topology, R is called the Sorgenfrey line.
   (a) Show that each of these basis sets is also closed in this topology.
   (b) Show that the Sorgenfrey line is normal. [Solution on p.198 Munkres]
   (c) Show that the Sorgenfrey line is separable, first countable, but not second countable. [Solution on p. 192 Munkres]
   (d) Show that the Sorgenfrey line is not metrizable.
2. Show that 'Tychonoff', 'metrizable', and 'locally compact' are topological properties. Show that 'Tychonoff' is a hereditary property.
3. Let X, Y be two (Hausdorff say) topological spaces. Show that two continuous functions from X to Y that are equal on a dense subset are equal everywhere.
4. (This question will not be graded.) By looking at a two or three point set, find a topology which is T_0 but not T_1? Also, find a second countable Hausdorff space that is not regular (or equivalently, not metrizable) [Hint: See p. 197 in Munkres.].
5. (a) Show that a subspace of a first (resp. second) countable space is again first (resp. second) countable.
   (b) Show that R^n is second countable.
   (c) Show that if (x_n) is a sequence in a compact subset K of a first countable space, then (x_n) has a subsequence which converges to a point in K.
   (d) Show that a subspace of a separable metric space is separable.
6. (a) If $f_1, f_2, \cdots, f_n$ are a finite family of continuous functions from X into $[0, \infty)$, prove that the function $f(x) = \max\{f_1(x), f_2(X), \cdots, f_n(x)\}$ is continuous.
   (b) If X is Tychonoff, and if E, F are disjoint closed sets in X, with E compact, show that there exists a continuous function $f : X \to [0, 1]$ which is 1 on E and 0 on F.
7. Show that if X is normal (resp. regular) then any pair of disjoint closed sets (resp. points) have neighborhoods whose closures are disjoint.
8. In this question $p : X \to Y$ is a continuous closed surjective map with $p^{-1}(\{y\})$ compact for every $y \in Y$. Show that:
   (a) If X is Hausdorff then so is Y.
   (b) If X is regular or normal then so is Y.
   (c) If X is locally compact then so is Y.
   (*d) If X is second countable then so is Y.
9. Recall that a G_δ set is one which is an intersection of a countable number of open sets. Prove the 'strong form' of Urysohn's lemma, namely that for disjoint closed G_δ sets A and B in a normal space X, there is a continuous function $g : X \to [0, 1]$ that is 1 precisely on A and 0 precisely on B (that is $0 < g < 1$ outside both A and B). [Hint: one could

first prove that a set A is a closed G_δ set if and only if there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in A$, and $f(x) > 0$ for $x \notin A$.]

10. Show that any subset of the rational numbers (with its usual topology) which contains a nonempty open set, cannot be compact. Deduce that the rationals is not locally compact.
11. Show that the one point compactification of $(0,1)$ is homeomorphic to the circle, giving all details.
12. Let X be a locally compact Hausdorff space, and let $a(X)$ be its 1-point compactification. Show that given any $f \in C_0(X)$, the function $\tilde{f}$ on $a(X)$ defined to be f on X and 0 at the 'adjoined point' α, is continuous on $a(X)$. Conversely, show that if $f \in C(a(X))$ with $f(\alpha) = 0$, then $f_{|X}$ is in $C_0(X)$.
13. Prove that the ordering between compactifications discussed in the final Remark in Section 2.4 is indeed an ordering, although 'transivity' here is subject to saying that two compactifications $(\hat{X}, j)$ and $(\tilde{X}, i)$ are 'equal' if there is a homeomorphism $f : \tilde{X} \to \hat{X}$ with $f \circ i = j$. Show also that the f there is unique, and is surjective. Show that the 1-point compactification is the smallest compactification in this ordering.
14. Show that a locally compact space K is metrizable and σ-compact, iff K is second countable.