Department of Mathematics, University of Houston
Math 6342. Topology. David Blecher
Homework 8

1. Show that a union of a family of connected subsets (resp. path connected subsets), no two of which are mutually disjoint, is connected (resp. path connected).

2. Show that a product of path connected spaces (with the product topology) is path connected.

*3. If X is compact and Hausdorff show that the (connected) component containing a point x is the intersection of all clopen sets containing x. [See e.g. Hint p. 236 in text].

4. Prove that if U is an open path connected set in R^n then U is polygonally connected (that is, for every $\vec{x}, \vec{y} \in U$, there exists a path in U from $\vec{x}$ to $\vec{y}$ made up of a finite number of straight line segments).

5. Prove that for nonempty sets in $\mathbb{R}$, the following things are all the same: connected, path connected, polygonally connected, convex, being an interval. We recall that an interval is a set I in $\mathbb{R}$ with the property $x, y \in I, x < z < y$ implies that $z \in I$; prove that these are the usual sets of one of the following 9 types:
$[a, b], [a, b), (a, b], (a, b), [a, \infty), (a, \infty), (-\infty, b], (-\infty, b), (-\infty, \infty)$.

6. A topological space X is called *locally Euclidean* if every point $x \in X$ has an open neighborhood which is homeomorphic to an open set in R^m, for some integer m. We also call this being m-*locally Euclidean*. Prove that a locally Euclidean space is locally compact, locally connected, and locally path connected.

7. A topological space X is called a *manifold* or an m-*manifold* if it is Hausdorff, second countable, and m-locally Euclidean (see question 6).
   (a) Show that any m-manifold is regular and metrizable.
   (b) A 2-manifold is called a *surface*, a 1-manifold is called a *curve*. Show that projective 2-space P^2 (see 1.5.12 Example 3 (v)) is a surface.
   (c) Show that every compact m-manifold is the topological sum of a finite number of connected compact m-manifolds. Recall that the topological sum is the 'disjoint union' of topological spaces X_i, and a set is open in the topological sum if and only if its intersection with each X_i is open in X_i.

8. (a) Show that if X is a compact Hausdorff m-locally Euclidean topological space (see question 6), then here is a topological imbedding from X into $\mathbb{R}^N$ for some N (This is the proof of Theorem 36.2 Munkres).
   (b) Deduce that X in (a) is second countable, hence is a compact m-manifold (see question 7).