Department of Mathematics, University of Houston
Topology - Blecher
Mid Term Exam

Instructions: Put all your bags and papers on the side of the room. Answer as many questions as you can. You may quote freely any results from the notes without proof, except those you are asked to prove.

1. (a) What is a directed set? What is a net in a set X?
   (b) What does it mean to say that a net converges?
   (c) Prove that if A is a non-empty subset of a topological space (X, τ), then $x \in \overline{A}$ iff there exists a net in A with limit x. Deduce from this that a set A is closed if whenever (x_t) is a net in A with $x_t \to x \in X$, then $x \in A$.
   (d) Prove that if $f : X \to Y$ is continuous and $x_t \to x$ in X, then $f(x_t) \to f(x)$.

2. (a) Show that any collection $\mathcal{A}$ of subsets of a set X, such that $\cup\{A : A \in \mathcal{A}\} = X$, is the subbasis for a topology on X. What is the basis for this topology (and prove it is a basis)?
   (b) If E is a subset of a topological space (X, τ), what is the subspace topology of E?
   (c) Let (X, τ) be a topological space, and suppose that Y is a subset of X, endowed with the subspace topology τ_Y. Prove that a net in Y is convergent to a point in Y with respect to the topology τ if and only if it is convergent to that point with respect to the subspace topology τ_Y.
   (d) If $f : X \to Y$ is continuous, $A \subset X$, show that $f|_A$ is continuous on A (the latter with its subspace topology).

3. (a) How is a quotient topology on $X/\sim$ defined?
   (b) What is a *quotient map* between topological spaces?
   (c) In what way are quotient topologies and quotient maps the same? Describe this in detail, and then prove it.
   (d) Prove that the labelling $aba^{-1}b^{-1}$ gives rise to a quotient space homeomorphic to the torus $S^1 \times S^1$ (the surface of a donut). Here S^1 is the circle $x^2 + y^2 = 1$.

4. (a) Define an 'initial topology'. Also write down a basis for such a topology.
   (b) State and prove a criterion for a net to converge in an initial topology.
   (c) Define the product topology on $\prod_{j\in J} X_j$.
   (d) Complete the sentence: "A net x_λ in the product topology on $\prod_{j\in J} X_j$ converges if and only if ...". Also say why this is true.
   (e) Describe a basis for the product topology, and prove that this is a basis.
5. (a) What is a compact set in a topological space? Also, state several conditions equivalent to the definition.
   (b) Explain why a compact subset of a Hausdorff space is closed.
   (c) Show that the intersection of any family of compact subsets in a Hausdorff topological space X is compact.
   (d) What is the Tychonoff theorem?
   (e) Prove that if $f : X \to Y$ is continuous, one-to-one and onto, and if X is compact, and Y is Hausdorff, then f is a homeomorphism.
6. (a) State Urysohn's lemma.
   (b) Define 'completely regular' (a.k.a. 'Tychonoff'/T 3 1/2).
   (c) What is a 'hereditary property'?
   (d) Prove that 'T 3 1/2' is a hereditary property.