Department of Mathematics, University of Houston
Topology - Blecher
Mock Final Exam .

Instructions: Put all your bags and papers on the side of the room. Answer three full questions. If you have time, a 4th question may then be attempted, but it will be graded for halved point values (if you do this write '4th question next to it'). Any other questions attempted will not be graded. [The real final may not have so much choice, or may require you to answer questions on several different parts of the course, or may look quite different to this mock final.]

1. (a) Show that two continuous functions between Hausdorff spaces that are equal on a dense subset are equal everywhere.
   (b) Use (a) to show that the Stone-Cech compactification of a space X is unique: that is if $(\hat{X}, k)$ was another compactification of X with the same universal property (that is, for any compact Hausdorff space C, and for any continuous $f : X \to C$, there exists continuous $\tilde{f} : \hat{X} \to C$ such that $\tilde{f} \circ k = f$), then there exists a homeomorphism $g : \beta X \to \hat{X}$ such that $g \circ j = k$.
   (c) Define the Cantor set.
   (d) Show that the Cantor set is compact, has no isolated points or interior, and is homeomorphic to a countable product of copies of $\{0, 1\}$.
2. (a) What is a T$3\frac{1}{2}$ space (also known as 'completely regular' or 'Tychonoff')? What is a cube $[0, 1]^A$?
   (b) State the Urysohn lemma and the imbedding lemma.
   (c) Show that T$3\frac{1}{2}$ is a hereditary property.
   (d) Show that every T$3\frac{1}{2}$ space is homeomorphic to a subspace of a cube.
3. (a) Define: second countable, separable.
   (b) Show that second countable implies separable. Show also that 'second countable' is a hereditary property.
   (c) State a 4-part characterization of separable metrizable spaces.
   (d) Using (c), prove that a compact Hausdorff space is metrizable if and only if it is second countable.
4. (a) What is a locally compact space? State three or four conditions that are equivalent to being locally compact and Hausdorff.
   (b) Show that the rationals (with their usual topology) is not locally compact.

(c) Is every locally compact space homeomorphic to an open subspace of a compact space? Prove it or give a counterexample.

(d) State and prove a Urysohn type lemma valid for all locally compact Hausdorff spaces.

5. (a) Define the one point (Alexandrov) compactification $\mathtt{a}(X)$.

   (b) Sketch the proof that every non-compact topological space (X, τ) has an Alexandrov compactification.

   (c) What is the relationship between $C_0(X)$ and $C(\mathtt{a}(X))$ for locally compact X? You dont need to prove it, except that part of the proof is in (d).

   (d) Let X be a locally compact Hausdorff space, and let $a(X)$ be its 1-point compactification. Show that given any $f \in C_0(X)$, the function $\tilde{f}$ on $a(X)$ defined to be f on X and 0 at the 'adjoined point' α, is continuous on $a(X)$. Conversely, show that if $f \in C(a(X))$ with $f(\alpha) = 0$, then $f_{|X}$ is in $C_0(X)$.

6. (a) Define: connected topological space, (connected) component.

   (b) Prove that every component is connected and closed.

   (c) State the Intermediate Value Theorem (IVT) as a statement about functions on a connected (connected and in addition compact) space.

   (d) Prove that open path connected sets in R^n are polygonally connected.

7. (a) Prove that if (X, τ) is a topological space, and if A and Y are subsets of X, with A clopen and Y connected, then either $Y \subset A$ or $Y \subset A^c$.

   (b) Define 'path connected' and show that every path connected topological space is connected.

   (c) Show that a connected set in the reals must be one of the 9 types of intervals. Give all details.

   (d) What can you say about the path components of a locally path connected topological space?